Radius of convergence of p-adic connections and the Berkovich ramification locus

by

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Abstract

We apply the theory of the radius of convergence of a p-adic connection [2] to the special case of the direct image of the constant connection via a finite morphism of compact p-adic curves, smooth in the sense of rigid geometry. We show that a trivial lower bound for that radius implies a global form, theorem 0.1, of Robert's p-adic Rolle theorem. The proof is based on a widely believed, although unpublished, result of simultaneous semistable reduction for finite morphisms of smooth p-adic curves. We deduce from it a useful description, theorem 0.8 and its corollary 0.11, of the Galois structure of étale coverings of smooth curves, which is new and of independent interest. We finally take this opportunity to clarify the relation between the notion of radius of convergence used in [2] and the more intrinsic one used by Kedlaya [19, Def. 9.4.7].

Contents

0 A global p-adic Rolle theorem

Let $(k, |\ |)$ be a complete algebraically closed extension of $(\mathbb{Q}_p, |\ |_p)$, with $|p|_p = p^{-1}$, and let k° be the ring of integers of k. We consider k-analytic spaces in the sense of Berkovich. We want to illustrate our theory of the radius of convergence of a p-adic connection [2], by deducing from it a conceptual proof of a global form of Robert's p-adic Rolle theorem [23, §2.4], [24, Prop. A.20]. Our result is weaker than Robert's, but indicates a new approach to the problem and, in favorable global situations, offers a finer geometric understanding.

Let $\varphi : Y \to X$ be a morphism of smooth k-analytic curves. If φ is étale at a k-valued point $y \in Y(k)$, then, as in the familiar complex case, φ induces an open embedding

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$\varphi_{|U} : U \hookrightarrow X$, of an open neighborhood U of y, in X. But, as is rather the case in algebraic geometry, this property may fail at a more general type of point $y \in Y$, even if φ is étale at y [14]. In general, the points at which φ is not a local open embedding form a closed subset $\mathcal{R}_\varphi \subset Y$, called the *Berkovich ramification locus.* So, $\mathcal{R}_\varphi \cap Y(k) = \mathrm{Crit}(\varphi)$, the set of *critical points* of φ, at which the map fails to be étale. We are interested in bounding from below the distance of $\mathcal{R}_\varphi$ from a non-critical k-valued point $y \in Y(k)$, in the case of a finite morphism φ, as above, of compact curves.

Our interest in this topic arose from reading Faber's papers [15] [16], where this question is answered, via explicit computations, for a non-constant rational function φ, viewed as a finite flat map $\varphi : \mathbf{P} \to \mathbf{P}$, of the k-analytic projective line $\mathbf{P}$ to itself. The novelty in Faber's paper concerns the case of an open disk $D \subset \mathbf{P}$, with $D \cap \mathrm{Crit}(\varphi) = \emptyset$, such that $\varphi(D) = \mathbf{P}$, a case which cannot be deduced from the classical statement. We cannot prove either Robert's or Faber's result completely with our method. We prove instead

Theorem 0.1. *Let $\varphi : Y \to X$ be a finite morphism of compact rig-smooth strictly k-analytic curves. Let $B(\varphi) = \varphi(\mathrm{Crit}(\varphi)) \subset X(k)$ be the* classical branch locus *of φ, and let $Z(\varphi) = \varphi^{-1}(B(\varphi)) \subset Y(k)$ be the (saturation of the)* classical ramification locus *of φ. Let $D \subset Y \setminus Z(\varphi)$ be any open disk equipped with a* normalized coordinate *$T : D \xrightarrow{\sim} D(0, 1^-)$. Then, for any open disk $D' \subset D$ of T-radius $\leq p^{-\frac{1}{p-1}}$, φ induces an open embedding $D' \to X$. Moreover, if φ is residually separable at the boundary point ζ of D in Y, then φ induces an open embedding $D \to X$ of D itself.*

Remark 0.2. A k-analytic curve E is an *open disk* if it is isomorphic to the open analytic domain

$$D(0, r^-) = \{x \in \mathbf{A} | |T(x)| < r\} ,$$

of the k-analytic T-line $\mathbf{A}$, for some $r \in \mathbb{R}_{>0}$. The only isomorphism invariant of E is then the image of r in $\mathbb{R}_{>0}/|k^\times|$. We say that the open disk E is a *strict*, if $r \in |k^\times|$, *i.e.* if E is isomorphic to the standard unit open disk $D(0, 1^-)$. An open disk which is a relatively compact analytic domain in a k-analytic curve X, is strict (resp. non strict) if and only if its boundary point in X is of type 2 (resp. 3). This topic will be clarified in [11]. In particular, the point $\zeta \in Y$ at the boundary of D in the statement 0.1, is a point of type 2. We also use the notation

$$D(0, r^+) = \{x \in \mathbf{A} | |T(x)| \leq r\} ,$$

for the standard closed disk in $\mathbf{A}$, and $\alpha_{0,r}$ for its maximal point.

Remark 0.3. Let $\xi = \varphi(\zeta)$ be as above. The map $\varphi : Y \to X$, $\zeta \mapsto \xi$, induces an isometric embedding $\mathscr{H}(\xi) \subset \mathscr{H}(\zeta)$, hence a $\widetilde{k}$-linear embedding $\widetilde{\mathscr{H}(\xi)} \subset \widetilde{\mathscr{H}(\zeta)}$. Then ξ is a point of type 2, as well. The degree of inseparability $[\widetilde{\mathscr{H}(\zeta)} : \widetilde{\mathscr{H}(\xi)}]_i$, is a power of p, called the *residual inseparability of φ at ζ*. Then φ is *residually separable at ζ* if its residual inseparability at ζ is 1. In particular, if φ is tame at ζ [4, 6.3], then it is residually separable at ζ.

Remark 0.4. The T-radius of D' appearing in the theorem, will be called the *relative radius of D' in D* or the *height* of the semi-open annulus $D \setminus D'$. It is an analytic invariant $0 < h(D \setminus D') \leq 1$ of $D \setminus D'$ as in [8, §2].

Remark 0.5. Let $\varphi : D(0, 1^-) \to D(0, 1^-)$ be a morphism of the open unit k-disk to itself, such that $\varphi(0) = 0$, while $\varphi(D(0, 1^-))$ is not contained in $D(0, \rho^-)$, for any $\rho < 1$. Then φ is surjective. This follows from the elementary theory of Newton polygons. In fact, in the standard coordinate T, φ is represented by a power series $\varphi(T) = \sum_{i=r}^\infty a_i T^i$, with $a_i \in k^\circ$, such that $a_r \neq 0$ for some $r \geq 1$, and $\inf_i \mathrm{ord}_p a_i = 0$. So, for any $a \in D(0, 1^-)$, $\mathrm{ord}_p a > 0$,

the Newton polygon of $\varphi(T) - a$ has a side with negative slope $-\sigma$, where

$$\sigma := \frac{\mathrm{ord}_p a - \mathrm{ord}_p a_{r+j}}{r+j} > 0 \ ,$$

for some $j \geq 0$. See Fig. 1.

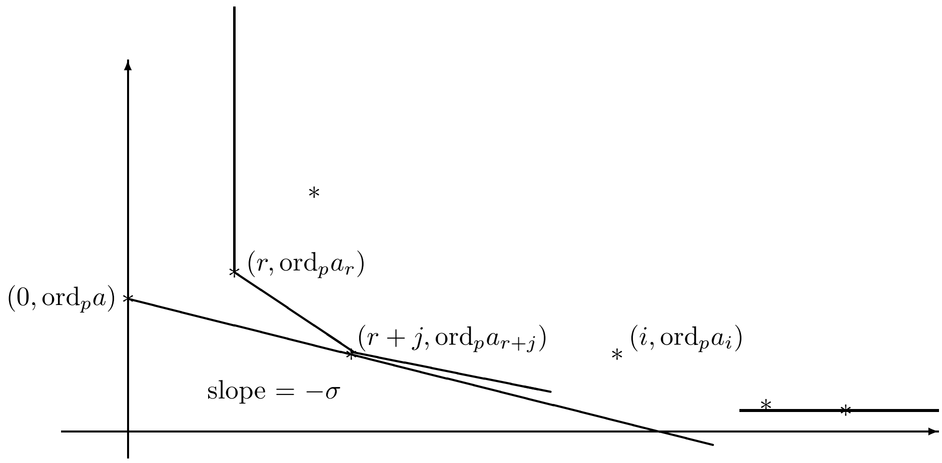

Figure 1: The Newton polygon of $\varphi(T) - a$

Remark 0.6. The classical p-adic Rolle theorem states that if $\varphi : D(0, 1^-) \to \mathbf{A}$ is any étale morphism of the open unit disk to the k-analytic affine line $\mathbf{A}$, then the restriction of φ to any open disk of radius $p^{-\frac{1}{p-1}}$ is an open embedding. Our result may be deduced from the classical one, as follows. We assume, with no loss of generality, that X and Y are connected. Suppose first that X is *projective.* If the genus of X is ≥ 1, it follows from [3, 4.5.3] that $\varphi(D)$ is contained in an open disk contained in X. So, the classical theorem applies. The case when X is the projective line and $\varphi_{|D}$ is not surjective, is covered by the classical theorem, too. If $\varphi(D) = X = \mathbf{P}$, then, φ being finite, the p-adic GAGA implies that Y is projective as well. But then the assumption on the branch locus is only verified if φ is an isomorphism, which contradicts $\varphi(D) = \mathbf{P}$. Now, (*cf.* [12, 3.2]) a compact rig-smooth curve is either affinoid or projective. But we know [2, 1.2.5] that, if X is affinoid, then it is an affinoid domain in a connected projective curve C, *formal* with respect to a strictly semistable model $\mathfrak{C}$ of C. So, again, [3, 4.5.3] shows that the only case not covered by the classical theorem is when $X = \mathbf{P}$ and φ is surjective, which leads us back to the former discussion.

Moreover, the classical theorem does not assume the existence of a compactification of the morphism as in our statement.

Remark 0.7. The rational function $\varphi(T) = \frac{T^{p+1}-p}{T}$ restricts to a surjective étale map $D(0, 1^-) \to \mathbf{P}$ to which the classical theorem does not apply, but Faber's does. This is example 5.3 in [16]. Notice that $0 \in Z(\varphi) \setminus \mathrm{Crit}(\varphi)$, so that our statement does not apply.

Our proof is based on the most basic result on p-adic differential systems, namely the so-called *trivial estimate* for the radius of convergence of their solutions [13, p. 94], once a certain integrality result is established. We deduce this integrality statement from a result on simultaneous semistable reduction of k-analytic curves, (3.4) below, due to Coleman [10] and improved by Liu [21] in the projective case. See also Temkin [26]. This result is apparently well-known to specialists, but, as far as we know, unpublished.

We deduce from our integrality statement the following result, of independent general interest.

Theorem 0.8. *Let $\varphi : Y \to X$ be a finite morphism of connected compact rig-smooth strictly k-analytic curves and let $D \subset Y$ be a strict open disk such that $\varphi(D)$ is not compact. Then $\varphi(D) =: E$ is a strict open disk in X. Assume the induced covering $\varphi^{-1}(E) \to E$ is étale. Then $\varphi^{-1}(E) \to E$ is Galois. In particular,*

$$\varphi^{-1}(E) = \dot{\bigcup}_{i=1}^{r} D_i \ ,$$

with $D = D_1$, a disjoint union of open disks isomorphic to D, such that the induced covering $\varphi_i : D_i \to E$ is Galois with group the stabilizer of D_i.

Remark 0.9. As we observed before, if $\varphi(D)$ is compact, then $\varphi(D) = X \cong \mathbf{P}$, and therefore, if φ is unramified, it is an isomorphism.

Remark 0.10. We observed in remark 0.6 that if $\varphi(D)$ is not compact, then it is contained in a strict open disk in X. From remark 0.5 it then follows that $\varphi(D) = E$ is a strict open disk in X.

We deduce from (0.8) the following remarkable consequence.

Corollary 0.11. *Any finite étale covering of an open disk by an open disk is Galois.*

Proof. Suppose $\varphi : D(0, 1^-) \to D(0, 1^-)$ is a finite étale covering of degree d, such that $\varphi(0) = 0$. It will suffice to prove that for any $\rho \in |k| \cap (0, 1)$, the restriction of φ to $\varphi_\rho : \varphi^{-1}(\varphi(D(0, \rho^-)) \to D(0, \rho^-)$ is Galois. Let

$$R(\rho) = \sup\{r \in (0, 1) | \varphi(D(0, r^-)) \subset D(0, \rho^-)\} \ .$$

We have $R(\rho) < 1$, because $\varphi(D(0, 1^-)) = D(0, 1^-)$. Moreover, since $\varphi(D(0, r^-))$ is an open disk in $D(0, 1^-)$, as we saw before, we have $\varphi(\alpha_{0,R(\rho)}) = \alpha_{0,\rho}$. In particular, $\alpha_{0,R(\rho)}$ is a point of type 2, so that $R(\rho) \in |k| \cap (0, 1)$ and $\varphi(D(0, R(\rho)^-) = D(0, \rho^-)$. Let $\varphi^+_\rho : \varphi^{-1}(\varphi(D(0, \rho^+)) \to D(0, \rho^+)$ be the restriction of φ. This map, together with the inclusion $D(0, R(\rho)^-) \subset \varphi^{-1}(\varphi(D(0, \rho^+))\ (\subset D(0, 1^-))$, satisfies the assumptions of theorem 0.8. We conclude that $\varphi_\rho : \varphi^{-1}(\varphi(D(0, \rho^-)) \to D(0, \rho^-)$ is Galois. □

Although not strictly needed for the conclusion of our proof (a reference to either one of [2] or [19], independently, would suffice), we recall in the last section the main properties of the radius of convergence of a connection on a compact rig-smooth p-adic analytic curve X with poles at a finite subset $Z \subset X(k)$ [2]. In that paper, we consider a (sufficiently fine) strictly semistable k°-formal model $\mathfrak{X}$ of X and an extension $\mathfrak{Z}$ of Z to a divisor of the smooth part of $\mathfrak{X}$, étale over k°. We then introduce a *global* notion of $(\mathfrak{X}, \mathfrak{Z})$*-normalized radius of convergence* $\mathcal{R}_{\mathfrak{X},\mathfrak{Z}}(x, (\mathcal{M}, \nabla))$ *of* $(\mathcal{M}, \nabla) \in \mathbf{MIC}((X \setminus Z)/k)$ *at* $x \in X \setminus Z$. We take this opportunity to completely clarify the relation between $\mathcal{R}_{\mathfrak{X},\mathfrak{Z}}(x, (\mathcal{M}, \nabla))$ and the *local* notion of *intrinsic generic radius of convergence* $IR(\mathcal{M}_{(x)}, \nabla)$ *of* $(\mathcal{M}, \nabla)$ *at* x, for a point $x \in X$ of Berkovich type 2 or 3, used by Kedlaya [19, Def. 9.4.7]. The coincidence of the two notions when x is a point of the skeleton $\Gamma_{\mathfrak{X},\mathfrak{Z}} \setminus Z$, should be useful in general. It is here only used implicitly (in an obvious case) in the conclusion of our proof.

I am indebted to V. Berkovich and to X. Faber for their explanations on the p-adic Rolle theorem and to R. Coleman, Q. Liu, M. Raynaud and M. Temkin for help in the statement of theorem 3.4. Discussions with P. Berthelot, M. Cailotto and Q. Liu have been most useful in the preparation of this paper: I thank them heartly for that. It is a pleasure to aknowledge the well-founded criticism and the invaluable suggestions provided by the referee.

1 A change in viewpoint

1.1. Let notation be as in theorem 0.1. We use by default (strictly) k-analytic spaces in the sense of Berkovich [4] endowed with their natural topology. We assume without loss of generality that X and Y are connected strictly k-analytic curves. By (a minor variation of) [12, Cor. 3.4], they are good strictly k-analytic spaces. Now, a finite morphism of good analytic spaces is good and closed, so that proposition 3.2.9 of [4] applies, and φ is finite flat and, in particular, open. It follows that, for any strict affinoid $U \subset X$, $\varphi^{-1}U$ is affinoid and $\varphi_* \mathcal{O}_{\varphi^{-1}U}$ is locally free for the Zariski topology of U. Then, φ being good, it follows that $\varphi_* \mathcal{O}_Y$ is locally free for the natural topology of X. Since X is connected, the degree of φ is a constant d on X. Moreover, φ is generically étale (*i.e.* étale but at a discrete set of k-rational points) because we are in characteristic zero. We recall that an irreducible compact k-analytic curve is either the analytification of a projective curve or it is affinoid [17], [12, Prop. 3.2]. From the discussion of remarks 0.9 and 0.10 it follows that we may, and will, assume all over this paper, that $E := \varphi(D)$ is a strict open disk in X. We will denote by ζ (resp. ξ) the boundary point of D in Y (resp. of E in X).

1.2. Let $B = B(\varphi) \subset X(k)$ and $Z = Z(\varphi)$ be as before. Let $\mathcal{J}_B$ (resp. $\mathcal{J}_Z$) denote the ideal sheaf of B (resp. Z). For a coherent $\mathcal{O}_X$-module (resp. $\mathcal{O}_Y$-module) $\mathcal{F}$, we denote by $\mathcal{F}(*B)$ (resp. $\mathcal{F}(*Z)$) the union $\bigcup_{N \geq 1} \mathcal{J}_B^{-N} \otimes \mathcal{F}$ (resp. $\bigcup_{N \geq 1} \mathcal{J}_Z^{-N} \otimes \mathcal{F}$).

The map φ restricts to an étale covering $Y \setminus Z \to X \setminus B$ of degree d. Hence, $\Omega^1_{Y \setminus Z} = \varphi^* \Omega^1_{X \setminus B}$ and $\varphi_* \Omega^1_{Y \setminus Z} = \varphi_* \mathcal{O}_{Y \setminus Z} \otimes_{\mathcal{O}_{X \setminus B}} \Omega^1_{X \setminus B}$, by the projection formula. More precisely, $\Omega^1_Y(*Z) = \varphi^*(\Omega^1_X(*B))$ and $\varphi_*(\Omega^1_Y(*Z)) = \varphi_* \mathcal{O}_Y \otimes_{\mathcal{O}_X} \Omega^1_X(*B)$. The direct image

$$(1.2.1) \qquad \varphi_*(d_{Y/k} : \mathcal{O}_Y \to \Omega^1_Y) \ , \quad \text{that is} \quad \varphi_*(d_{Y/k}) : \varphi_* \mathcal{O}_Y \to \varphi_* \mathcal{O}_Y \otimes \Omega^1_X(*B) \ ,$$

is then a connection on the locally free $\mathcal{O}_X$-module $\mathcal{F} := \varphi_* \mathcal{O}_Y$ of rank d, with poles at B. We denote by $\mathbf{MIC}(X(*B)/k)$ the tannakian category of such objects, so that

$$(1.2.2) \qquad (\mathcal{F}, \nabla_{\mathcal{F}}) := (\varphi_* \mathcal{O}_Y, \varphi_*(d_{Y/k})) \in \mathbf{MIC}(X(*B)/k) \ .$$

Notice that $\varphi_* \mathcal{O}_Y$ is also a sheaf of commutative $\mathcal{O}_X$-algebras and that the multiplication map

$$(1.2.3) \qquad \mu_Y : \varphi_* \mathcal{O}_Y \otimes_{\mathcal{O}_X} \varphi_* \mathcal{O}_Y \longrightarrow \varphi_* \mathcal{O}_Y$$

is horizontal.

1.3. We define two sheaves on X. The first

$$(1.3.1) \qquad \mathcal{S}ect(Y/X) := \mathscr{H}om_{\mathcal{O}_X-\mathrm{alg}}(\varphi_* \mathcal{O}_Y, \mathcal{O}_X) \ ,$$

is a sheaf of finite sets of cardinality $\leq d$. It is the *sheaf of local sections* of Y/X or of φ. In fact, for any affinoid $U \subset X$, $V := \varphi^{-1}(U) \subset Y$ is an affinoid domain as well, and

$$Hom_X(U, Y) = Hom_U(U, V) = Hom_{\mathcal{O}(U)-\mathrm{alg}}(\mathcal{O}(V), \mathcal{O}(U))$$

$$= \Gamma(U, \mathscr{H}om_{\mathcal{O}_X-\mathrm{alg}}(\varphi_* \mathcal{O}_Y, \mathcal{O}_X)) = \Gamma(U, \mathcal{S}ect(Y/X)) \ .$$

The second is the sheaf of k-vector spaces of dimension $\leq d$

$$(1.3.2) \qquad \mathcal{S}ol(\mathcal{F}, \nabla_{\mathcal{F}}) := \mathscr{H}om_{\mathcal{O}_X}((\mathcal{F}, \nabla_{\mathcal{F}}), (\mathcal{O}_X, d_{X/k}))^{\nabla} \ ,$$

called the *sheaf of local solutions* of $(\mathcal{F}, \nabla_{\mathcal{F}})$. Notice that for any $x_0 \in X(k) \setminus B$, there exists an open neighborhood U of x_0, such that $\mathcal{S}ect(Y/X)_{|U}$ is the constant sheaf $\{1, \dots, d\}$ and that $\mathcal{S}ol(\mathcal{F}, \nabla_{\mathcal{F}})_{|U}$ is a k-local system of rank d.

The crucial remark is

Lemma 1.4. *We have an inclusion of sheaves of sets*

$$\mathcal{S}ect(Y/X) \subset \mathcal{S}ol(\mathcal{F}, \nabla_{\mathcal{F}}) \ . \tag{1.4.1}$$

For any $x \in X \setminus B$, $\mathcal{S}ect(Y/X)_x$ is a k-basis of $\mathcal{S}ol(\mathcal{F}, \nabla_{\mathcal{F}})_x$, i.e. *the sheaf of k-vector spaces $\mathcal{S}ol(\mathcal{F}, \nabla_{\mathcal{F}})$ is freely generated by its subsheaf $\mathcal{S}ect(Y/X)$.*

Proof. We observe that the construction

$$(\varphi : Y \to X) \quad \longmapsto \quad (\varphi_* \mathcal{O}_Y, \varphi_*(d_{Y/k}), \mu_Y) \ , \tag{1.4.2}$$

from finite coverings to finite locally free $\mathcal{O}_X$-algebras with a connection and horizontal multiplication map, is functorial [1, App. E]. But $\varphi : Y \to X$ is determined by the $\mathcal{O}_X$-algebra $(\varphi_* \mathcal{O}_Y, \mu_Y)$ alone. As a consequence, for any affinoid domain $U \subset X$, any $\mathcal{O}_U$-algebra homomorphism $\varphi_*(\mathcal{O}_Y)_{|U} \to \mathcal{O}_U$, is automatically horizontal, hence a solution of $(\mathcal{F}, \nabla_{\mathcal{F}})_{|U}$. This proves the first part of the lemma.

As for the second part of the statement, since two sections of $\varphi_* \mathcal{O}_Y$ on a connected affinoid domain U coincide as soon as they coincide in the neighborhood of a k-rational point of U, it will suffice to treat the case of $x \in X(k)$. So, for any point $x_0 \in X(k) \setminus B$, we consider the completion $\widehat{\mathcal{O}}$ of the local ring $\mathcal{O}_{X,x_0}$ and its formal spectrum $\widehat{X} = \mathrm{Spf}\, \widehat{\mathcal{O}}$; it is a formal power series ring of the form $k[[t]]$, where t is a local parameter at x_0, which we may assume to extend to a section of $\mathcal{O}_X$. We informally denote by $W \mapsto \widehat{W}$ the base-change functor by $\widehat{X} \to X$ on objects W defined over X. It will be enough to prove the statement for the map $\widehat{\varphi} : \widehat{Y} \to \widehat{X}$, at any $x_0 \in X(k) \setminus B$.

Notice that

$$\mathcal{S}ect(Y/X)\widehat{\ } = \mathscr{H}om_{\mathcal{O}_{X,x_0}-\mathrm{alg}}((\varphi_* \mathcal{O}_Y)_{x_0}, \mathcal{O}_{\widehat{X}}) = \mathscr{H}om_{\mathcal{O}_{\widehat{X}}-\mathrm{alg}}(\widehat{\varphi_* \mathcal{O}_Y}, \mathcal{O}_{\widehat{X}}) \ ,$$

is the set of formal sections of φ at x_0. The $\mathcal{O}_{\widehat{X}}$-algebra $\widehat{\mathcal{F}} = \widehat{\varphi_* \mathcal{O}_Y}$ is a direct sum

$$\widehat{\mathcal{F}} = \bigoplus_{i=1}^{d} \mathcal{O}_{\widehat{X}}\, e_i \ ,$$

where the e_i's are orthogonal idempotents. An algebra homomorphism $\sigma : \widehat{\varphi_* \mathcal{O}_Y} \to \mathcal{O}_{\widehat{X}}$, is forced to map one of the e_i's to 1, and the others to 0 : let us denote it by e_i^*. As we saw before, the e_i^*, for $i = 1, \dots, d$ are horizontal. Since they freely span the $\mathcal{O}_{\widehat{X}}$-module $\mathscr{H}om_{\mathcal{O}_{\widehat{X}}}(\widehat{\varphi_* \mathcal{O}_Y}, \mathcal{O}_{\widehat{X}})$ of rank d, they form a k-basis of $\mathscr{H}om_{\mathcal{O}_{\widehat{X}}}(\widehat{\varphi_* \mathcal{O}_Y}, \mathcal{O}_{\widehat{X}})^{\nabla} = \mathcal{S}ol(\mathcal{F}, \nabla_{\mathcal{F}})_x$. This proves the statement. □

1.5. Our problem consists in the determination of the maximal open disk $D_{y_0} \subset D$, centered at $y_0 \in Y(k) \cap D$, such that the map φ restricts to an isomorphism

$$D_{y_0} \xrightarrow{\ \sim\ } D'_{\varphi(y_0)} \ , \tag{1.5.1}$$

where $D'_{\varphi(y_0)}$ denotes an open disk with $\varphi(y_0) \in D'_{\varphi(y_0)} \subset X$.

If we set $x_0 = \varphi(y_0) \in X(k) \setminus B$, this problem coincides with the problem of determining the maximal open disk D'_{x_0}, centered at x_0, such that the unique local section σ of φ at x_0 such that $\sigma(x_0) = y_0$ converges on D'_{x_0}. By lemma 1.4, σ is a local solution of $(\mathcal{F}, \nabla_{\mathcal{F}})$ at x_0. Notice that we will then need to express the result not in terms of D'_{x_0}, but in terms of the height of the annulus $D \setminus D_{y_0}$ in (1.5.1), where $D_{y_0} = \sigma(D'_{x_0})$. The statement we want to prove says that

$$h(D \setminus D_{y_0}) \geq p^{-\frac{1}{p-1}} \ .$$

Obviously, in this discussion Y may be replaced by any compact strictly k-analytic domain $C \subset Y$, and X by the image $\varphi(C) \subset X$, provided $D \subset C$ and φ induces a finite morphism $C \to \varphi(C)$.

1.6. I am indebted to Liu for the following general statement

Proposition 1.7. *Let $\varphi : Y \to X$ be a finite morphism of k-analytic spaces. Let $\mathcal{G}$ be any coherent $\mathcal{O}_Y$-module. Then*

$$\varphi^*\varphi_*\mathcal{G} = \varphi^*\varphi_*\mathcal{O}_Y \otimes_{\mathcal{O}_Y} \mathcal{G} \ .$$

The proof is immediate and follows the footprints of the proof of the analogous algebraic statement, namely

Proposition 1.8. *Let $\varphi : Y \to X$ be an affine morphism of k-schemes. Let $\mathcal{G}$ be any quasi-coherent $\mathcal{O}_Y$-module. Then*

$$\varphi^*\varphi_*\mathcal{G} = \varphi^*\varphi_*\mathcal{O}_Y \otimes_{\mathcal{O}_Y} \mathcal{G} \ .$$

So, going back to the notation of this paper, we observe that

$$\varphi^*\varphi_*(d_{Y/k} : \mathcal{O}_Y \to \Omega^1_Y) = \varphi^*\varphi_*(d_{Y/k}) : \varphi^*\varphi_*\mathcal{O}_Y \to \varphi^*\varphi_*\mathcal{O}_Y \otimes \Omega^1_Y \ . \tag{1.8.1}$$

In particular,

Corollary 1.9. *The inverse image connection*

$$\varphi^*(\mathcal{F}, \nabla_{\mathcal{F}}) =: (\mathcal{E}, \nabla_{\mathcal{E}}) \in \mathbf{MIC}(Y/k) \ , \tag{1.9.1}$$

has no singularity on Y.

We also consider the fiber product $Y \times_X Y$ and its two projections $\mathrm{pr}_1, \mathrm{pr}_2 : Y \times_X Y \to Y$.

We have, as before:

Corollary 1.10. *The inverse image sheaf $\varphi^{-1}\mathcal{S}ect(Y/X)$ coincides with the sheaf $\mathcal{S}ect(\mathrm{pr}_1)$ of sections of $\mathrm{pr}_1 : Y \times_X Y \to Y$. We have an inclusion of sheaves of sets*

$$\varphi^{-1}\mathcal{S}ect(Y/X) \subset \mathcal{S}ol(\mathcal{E}, \nabla_{\mathcal{E}}) \ . \tag{1.10.1}$$

The sheaf of k-vector spaces $\mathcal{S}ol(\mathcal{E}, \nabla_{\mathcal{E}})_{|Y \setminus Z}$ is freely generated by its subsheaf $\varphi^{-1}\mathcal{S}ect(Y/X)_{|Y \setminus Z}$.

We will prove

Proposition 1.11. *The connection $(\mathcal{E}, \nabla_{\mathcal{E}})$ restricts to the trivial connection on any open disk contained in $Y \setminus Z$.*

Corollary 1.12. *The étale covering $\mathrm{pr}_1 : Y \times_X Y \to Y$ is trivial over any open disk $D \subset Y \setminus Z$. In particular, theorem 0.8 follows.*

Proof. (Of corollary 1.12) As we explained in 1.1, we may assume that $E = \varphi(D)$ is an open disk in X. Let $y = y_1 \in D(k)$, $x = \varphi(y)$, and let $\{y_1, \ldots, y_d\} = \varphi^{-1}(\{x\})$. So, $\mathrm{pr}_1^{-1}(\{y_i\}) = \{(y_1, y_1), \ldots, (y_1, y_d)\}$ consists of d distinct points, and the section τ_i of pr_1 such that $\tau_i(y_1) = (y_1, y_i)$ extends, by corollary 1.10 and proposition 1.11, to a solution of v_i of $(\mathcal{E}, \nabla_{\mathcal{E}})$ on D, which is a section of pr_1 in a neighborhood of y_1, hence all over D. The composition $\sigma_i := \mathrm{pr}_2 \circ v_i$ is therefore an open embedding $D \hookrightarrow Y$, such that the diagram

$$\begin{tikzcd} D \arrow[rr, "\sigma_i"] \arrow[dr, "\varphi"'] & & Y \arrow[dl, "\varphi"] \\ & X & \end{tikzcd} \tag{1.12.1}$$

commutes. In particular, σ_1 is the inclusion $D \hookrightarrow Y$. Then $\sigma_i(D)$, which cannot be compact under the assumption that $\varphi(D)$ is an open disk, is an open disk in Y. If $\sigma_i(D) \cap \sigma_j(D) \neq \emptyset$, we have $\sigma_i(D) = \sigma_j(D)$. In fact, the union $\sigma_i(D) \cup \sigma_j(D)$ cannot be compact, otherwise $\varphi(D)$ would also be compact, against our assumption. So, [2, Lemma 1.5.6] $\sigma_i(D) \cup \sigma_j(D)$ is the biggest of the two disks $\sigma_i(D)$ and $\sigma_j(D)$. We have $\tau_i(D) = D \times_E \sigma_i(D)$, for all i, and, since pr_1 induces an isomorphism $\tau_i(D) \xrightarrow{\sim} D$, we get a contradiction. So, now let $\{D = D_1, D_2, \dots, D_r\} = \{\sigma_1(D), \dots, \sigma_d(D)\}$ with $D_i \neq D_j$, for $i \neq j$. Then φ induces a finite étale covering of open disks $\varphi_i : D_i \to E$. Moreover, by our previous discussion where y_1 is possibly replaced by another $y_j \in \varphi^{-1}(\{x\})$, we see that, if $\{y_{i,1}, \dots, y_{i,e_i}\} = \varphi_i^{-1}(\{x\}$, with $y_{i,h} \neq y_{i,\ell}$ if $h \neq \ell$, then, for any choice of $h, \ell \in \{1, \dots, e_i\}$, there is precisely one covering transformation σ of $\varphi_i : D_i \to E$, such that $\sigma(y_{i,h}) = y_{i,\ell}$. In other words, φ_i is a Galois cover. Let G_i be the Galois group of φ_i. Now, for any $i = 1, \dots, r$, we let $\pi_i : D_1 \xrightarrow{\sim} D_i$ be such that $\pi_i(y_{1,1}) = y_{i,1}$. Then

$$G := \dot{\bigcup}_{i=1}^{r} G_i \circ \pi_i \ ,$$

is a group of covering transformations of $\varphi : \varphi^{-1}(E) \to E$, of degree $d = e_1 + \cdots + e_r$, which operates transitively on the fiber above x, hence on all fibers. □

Remark 1.13. The last statement in corollary 1.12 implies that the diagonal embedding $\Delta : Y \to Y \times_X Y$ is a local isomorphism at any $y \in Y \setminus Z$. It is then a more precise form, in the particular case we are considering, of [4, Prop. 3.3.7 *c*)].

Remark 1.14. The k-vector space of k-analytic solutions of $(\mathcal{E}, \nabla_{\mathcal{E}})$ at any point $z_0 \in Y(k) \backslash Z$ is spanned by the germs of analytic solutions $w(z)$ at $z = z_0$ of the algebraic equation $\varphi(w) = \varphi(z)$. Notice that if $\varphi : \mathbf{P} \to \mathbf{P}$ is a rational function, the algebraic equation for w as a function of z, $\varphi(z + w) = \varphi(z)$ coincides with the equation $w \cdot A_{\varphi}(z, w) = 0$ studied by Faber in section 2 of [16].

2 Basic results on p-adic differential systems

2.1. The classical theory of p-adic linear differential equations is developed on an open disk or an open annulus, embedded as open analytic domains in $\mathbf{P}$. Moreover, it is usually understood that their boundary points in $\mathbf{P}$ be points of Berkovich type 2. This precision becomes relevant when one insists that the coefficients of the equation represent germs of analytic functions at those boundary points. One classically defines, for $r \in (0,1) \cap |k|$, the k-Banach algebra $\mathscr{H}(r,1)$ of *analytic elements* [13, IV.4] on the open annulus

(2.1.1) $$C(0; r, 1) := \{x \in D(0, 1^-) \mid r < |T(x)| < 1\} \ .$$

It is the completion of the k-algebra of rational functions of T, with no poles within $C(0; r, 1)$, equipped with the sup-norm $|| \ ||$ on $C(0; r, 1)$. While $\mathscr{H}(r,1) \subset \mathcal{B}(r,1)$, the Banach k-algebra of bounded analytic functions on $C(r,1)$, the two do not coincide, and the properties of a first order system of linear differential equations

(2.1.2) $$\Sigma : \frac{dY}{dT} = GY \ ,$$

where G is a $n \times n$ matrix with coefficients in $\mathscr{H}(r,1)$ are more special than in the case of coefficients in $\mathcal{B}(r,1)$.

2.2. Let $C \subset Y$ be any open analytic domain, with only a finite set $\zeta_1, \ldots, \zeta_s$ of boundary points all of type 2 in Y. We define the Banach k-algebra $\mathscr{H}_Y(C)$ of Y*-analytic elements* on C as the completion of the k-algebra

$$\mathcal{O}_Y(C) \cap \bigcap_{i=1}^{s} \mathcal{O}_{Y,\zeta_i} \ ,$$

under the sup-norm $|| \ ||_C$ on C. In the present discussion, we have the open disk $D \subset Y$, with boundary point ζ of Berkovich type 2 (because of the assumption that D be isomorphic to $D(0, 1^-)$ [11]). We define a *formal coordinate* T on D in Y, to be a formal étale coordinate on (the smooth k°-formal model of) an affinoid domain $A \subset Y$, with good canonical reduction and maximal point ζ, which extends to an isomorphism $T : D \xrightarrow{\sim} D(0, 1^-)$. A formal coordinate T on D in Y is *overconvergent* if it extends as a section of $\mathcal{O}_Y$ on a neighborhood of ζ in Y.

Lemma 2.3. *Let D be a strict open disk in a compact rig-smooth strictly k-analytic curve Y. Then, a formal coordinate on D in Y exists.*

Proof. Let ζ be the boundary point of D in Y. Since ζ is of type 2, an affinoid domain $U' \subset Y$, with good canonical reduction and maximal point ζ, exists. If the open disk D intersects U', it is a residue class in U'; we set $U = U'$ in this case. Otherwise, $U' \cup D =: U$ is either an affinoid domain in Y, with good canonical reduction and maximal point ζ, or it coincides with Y, and Y is the analytification of a projective curve with good reduction. So, in any case $U = \mathfrak{U}_\eta$, for a smooth k°-formal scheme $\mathfrak{U}$, and $\mathrm{sp}_{\mathfrak{U}}(D) = a$ is a closed point of $\mathfrak{U}$. It suffices to pick an étale coordinate T on $\mathfrak{U}$ at a. □

Remark 2.4. With little more effort, one shows : if the strict open disks $D_1, \ldots, D_r \subset Y$ have the same boundary point in Y, a simultaneous formal coordinate on $D_1, \ldots, D_r$ in Y exists.

We now refine comma (3) of proposition 2.2.1 of [6] in our special case. Notice, contrarily to what is there stated, that the commas (2) and (3) of *loc.cit.* refer to points of type 3 and 2, respectively.

Lemma 2.5. *Notation as above. The point ζ admits a neighborhood in Y of the form $U = \mathscr{Y}_\eta^{\mathrm{an}} \setminus \coprod_{i=1}^n Y_i$, where $\mathscr{Y}$ is a connected smooth projective curve over k°, each Y_i is an affinoid domain isomorphic to the standard closed disk $D(0, 1^+)$, all of them in pairwise distinct residue classes of $\mathscr{Y}_\eta^{\mathrm{an}}$, with ζ the generic point of $\mathscr{Y}_s$ in $\mathscr{Y}_\eta^{\mathrm{an}}$. We may assume that the disk D is a residue class of $\mathscr{Y}_\eta^{\mathrm{an}}$ contained in U.*

Proof. We only have to prove the last part of the statement. The disk D intersects U in an open annulus B, with boundary point ζ. This annulus is connected and has the boundary point $\zeta \notin B$. It must then be the complement of a strict closed disk V in a single residue class C of $\mathscr{Y}_\eta^{\mathrm{an}}$. We construct a new connected compact smooth k-analytic curve Y', by glueing $D \subset Y$ to $\mathscr{Y}_\eta^{\mathrm{an}} \setminus V$, via B. The curve Y' is necessarily projective, since it is compact and it has empty Shilov boundary, and, since $Y' \setminus \{\zeta\}$ is a union of open disks, it has good reduction. So, $Y' = (\mathscr{Y}')_\eta^{\mathrm{an}}$, for a smooth projective curve $\mathscr{Y}'$ over k°. Notice that $D \cup U$ is now contained in Y', and that $U \cup D \cong \mathscr{Y}'_\eta \setminus \coprod_{i=1}^n Y'_i$, for Y'_i strict closed disks, contained in pairwise distinct residue classes of $(\mathscr{Y}')_\eta^{\mathrm{an}}$. □

Remark 2.6. With little more effort, we may prove : If $D_1, \ldots, D_r \subset Y$ are open disks with the same boundary point ζ in Y, there exists a neighborhood of ζ in Y, as in the lemma, such that $D_i \subset U$, for all $i = 1, \ldots, r$.

Corollary 2.7. *If the strict open disks $D_1, \ldots, D_r \subset Y$ have the same boundary point ζ in Y, a simultaneous formal overconvergent coordinate on $D_1, \ldots, D_r$ in Y exists.*

We will also need the following statement.

Proposition 2.8. *Let $\mathcal{M}$ be a locally free $\mathcal{O}_Y$-module of rank m, and let $D_1, \dots, D_r \subset Y$ be strict open disks with the same boundary point ζ in Y. Then, there exists a neighborhood U of ζ as produced in lemma 2.5 , such that $\mathcal{M}_{|U}$ is free.*

Proof. Let $U = \mathscr{Y}_\eta^{\mathrm{an}} \setminus \coprod_{i=1}^n Y_i$ be as in lemma 2.5, with $D_i \subset U$, for all i. Let C_i be the residue class of $\mathscr{Y}_\eta^{\mathrm{an}}$ containing Y_i. By increasing $Y_i \cong D(0,1^+)$ in C_i, we may assume that $\mathcal{M}$ is free over the open annulus $C_i \setminus Y_i$ (recall that $\mathcal{M}$ is locally free at ζ). Then we may use a basis of sections of $\mathcal{M}$ on $C_i \setminus Y_i$ to extend $\mathcal{M}$ to the residue class C_i. In the end we obtain a locally free coherent module $\overline{\mathcal{M}}$ on $\mathscr{Y}_\eta^{\mathrm{an}}$, which is then of the form $\mathscr{N}^{\mathrm{an}}$, for a locally free coherent $\mathcal{O}_{\mathscr{Y}_\eta}$-module $\mathcal{N}$. So, $\mathcal{N}$ is free on any open affine subset of $\mathscr{Y}_\eta$, and *a fortiori* $\mathcal{N}^{\mathrm{an}}$ is free on U. But $\mathcal{N}^{\mathrm{an}}_{|U} = \mathcal{M}_{|U}$. $\square$

Corollary 2.9. *Let $D \subset Y$ be as in lemma 2.3 and let ζ be the boundary point of D in Y. Let $\mathcal{M}$ be a locally free $\mathcal{O}_Y$-module of constant rank N. Then $\mathcal{M}(D) \cap \mathcal{M}_\zeta$ is a free module over $\mathcal{O}_Y(D) \cap \mathcal{O}_{Y,\zeta}$ of rank N. For any choice of a basis $\underline{v} := (v_1, \dots, v_N)$ of $\mathcal{M}(D) \cap \mathcal{M}_\zeta$ over $\mathcal{O}_Y(D) \cap \mathcal{O}_{Y,\zeta}$, let*

$$||\sum_{i=1}^N a_i v_i||_{\underline{v},C} = \max_i ||a_i||_D \ ,$$

be the corresponding norm on $\mathcal{M}(D) \cap \mathcal{M}_\zeta$. We define $(\mathscr{H}_Y^{(\underline{v})}(\mathcal{M}, D), || \ ||_{\underline{v},C})$ as the completion of $\mathcal{M}(D) \cap \mathcal{M}_\zeta$ under the norm $|| \ ||_{\underline{v},C}$. It is a Banach module over the k-Banach algebra $(\mathscr{H}_Y(D), || \ ||_D)$. For two choices $\underline{u}$ and $\underline{v}$ of an $\mathcal{O}_Y(D) \cap \mathcal{O}_{Y,\zeta}$-basis of $\mathcal{M}(D) \cap \mathcal{M}_\zeta$, the unique map of $(\mathscr{H}_Y(D), || \ ||_D)$-Banach modules

$$(\mathscr{H}_Y^{(\underline{u})}(\mathcal{M}, D), || \ ||_{\underline{u},C}) \to (\mathscr{H}_Y^{(\underline{v})}(\mathcal{M}, D), || \ ||_{\underline{v},C}) \ ,$$

sending u_i to v_i for all i, is a bounded isomorphism.

Definition 2.10. *We denote by $(\mathscr{H}_Y(\mathcal{M}, D), || \ ||_D)$ any representative of the uniquely defined isomorphism class of free finitely generated $(\mathscr{H}_Y(D), || \ ||_D)$-Banach modules defined by $(\mathscr{H}_Y(\mathcal{M}, D), || \ ||_{\underline{v},C})$, for any choice of $\underline{v}$.*

Let T be an overconvergent coordinate on D in Y. Then, for $C(0; r, 1)$ as in (2.1.1), we define $\mathscr{H}_{D,Y}(r, 1) := \mathscr{H}_Y(T^{-1}(C(0; r, 1)))$. In the particular case of $D = D(0, 1^-) \subset \mathbf{P}$, with canonical coordinate T, $\mathscr{H}(r, 1) = \mathscr{H}_{D,\mathbf{P}}(C(0; r, 1))$.

We assume that the entries of G in (2.1.2) are in $\mathscr{H}_{D,Y}(r, 1)$. Notice that

$$Frac(\mathscr{H}_Y(D)) \subset \bigcup_{r<1} \mathscr{H}_{D,Y}(r, 1) \ ,$$

so that if the entries of G are quotients of elements of $\mathscr{H}_Y(D)$, then the previous assumption is satisfied for $r < 1$, sufficiently close to 1. We let $t = T(\zeta) \in \mathscr{H}(\zeta)$. Notice that $g \mapsto |g(\zeta)|$ is a bounded multiplicative norm on $\mathscr{H}_{D,Y}(r, 1)$. The following (almost) classical definition will later be updated.

Definition 2.11. *The* generic radius of convergence *$R_{D \subset Y}(\Sigma)$ of the system Σ of (2.1.2) on $D \subset Y$, is defined by extending the field of constants from k to the valued field $\mathscr{H}(\zeta)$, so that the point ζ determines a canonical [2, Intro] $\mathscr{H}(\zeta)$-rational point $\zeta' \in Y \widehat{\otimes}_k \mathscr{H}(\zeta)$, such that $T(\zeta') = t$. Notice that the entries of G are analytic functions on the open disk of T-radius 1 in $Y \widehat{\otimes}_k \mathscr{H}(\zeta)$, centered at ζ', so that the system 2.1.2 is defined on that disk. Then $R_{D \subset Y}(\Sigma)$ is defined as the T-radius of the maximal open disk around ζ', of radius not exceeding 1, on which all solutions of Σ in $\mathscr{H}(\zeta)[[T - t]]$ converge.*

2.12. The number $R_{D\subset Y}(\Sigma)$ is computed as follows. We first iterate (2.1.2) into

$$\frac{1}{n!}\frac{d^n Y}{dT^n} = G_{[n]}\,Y \ , \tag{2.12.1}$$

and then

$$R_{D\subset Y}(\Sigma) = \min(1, \liminf_{i\to\infty} |G_{[i]}(\zeta)|^{-1/i}) \in (0,1] \ , \tag{2.12.2}$$

where the absolute value of a matrix is the maximum of the absolute values of its entries.

The generic radius of convergence of (2.1.2) is bounded below as follows [13, p. 94].

Proposition 2.13. (Trivial Estimate)

$$R_{D\subset Y}(\Sigma) \geq |G(\zeta)|^{-1}\, p^{-\frac{1}{p-1}} \ .$$

2.14. We now assume that the entries of the matrix G in (2.1.2) extend to meromorphic functions, necessarily with a finite number of zeros and poles, on the open disk $D = D(0,1^-)$. For example, this is the case if the entries of G will be in $Frac(\mathscr{H}_Y(D)$. We also assume that all singularities of the system Σ in $D(0,1^-)$ are *apparent* [13, V.5], *i.e.* that at any point $a \in D(0,1^-)(k)$, Σ admits a matrix solution in $GL(n, k((T-a)))$. Then the following *Transfer Theorem* in a disk with only apparent singularities, similar to [13, IV.5. A], holds.

Theorem 2.15. (Transfer Theorem) *Any solution of Σ at any k-rational point $x \in D(0,1^-)$ is meromorphic in a disk of T-radius $R_{D\subset Y}(\Sigma)$ around x.*

Proof. The point ζ induces on the field $k(T)$, of rational functions with coefficients in k in the overconvergent coordinate T, the classical Gauss norm $|\ |_{Gauss}$ [13]. Since the entries of G have a finite number of poles in D, we can follow the procedure of Proposition 5.1 of [13, Chap. V], to determine a $|\ |_{Gauss}$-unimodular matrix $P \in GL(n, k(T))$, such that $Y \mapsto PY$ transforms Σ into a system

$$\Sigma^{[P]} : \frac{dY}{dT} = G^{[P]}\,Y \ ,$$

with no singularities in $D(0,1^-)$. Notice that the entries of $G^{[P]}$ are then in $\mathscr{H}_Y(D)$. The matrix $P \in GL(n, k(T))$ is then ζ-*unimodular*, that is $|P(\zeta)| = |P^{-1}(\zeta)| = 1$. It follows that $|G^{[P]}_{[i]}(\zeta)| = |G_{[i]}(\zeta)|$, $\forall i$ and formula 2.12.2 shows that $R_{D\subset Y}(\Sigma^{[P]}) = R_{D\subset Y}(\Sigma)$. We may then assume from the beginning that Σ has no singularities in $D(0,1^-)$, that is thatthe entries of G are in $\mathscr{H}_Y(D)$. But then clearly $|G_{[i]}(x)| \leq |G_{[i]}(\zeta)| = ||G_{[i]}||_D$. A solution matrix of Σ at x is given by $Y_x(T) = \sum_{n=0}^{\infty} G_{[i]}(x)(T - T(x))^n$. So, $Y_x(T)$ converges for $|T - T(x)| < R_{D\subset Y}(\Sigma)$. □

2.16. A more precise version of the formula 1.2.1 is obtained if we view the pairs (Y,Z), (X,B) as smooth log-schemes over the log-field $(k, k^\times)$ [18]. The analytic map φ induces in fact a finite log-étale morphism $\varphi : (Y,Z) \to (X,B)$, locally free of degree d, so that

$$\Omega^1_Y(\log Z) = \varphi^* \Omega^1_X(\log B) \ .$$

Therefore formula 1.2.2 admits the refinement

$$\varphi_*(d_{Y/k} : \mathcal{O}_Y \to \Omega^1_Y(\log Z)) = \nabla_{\mathcal{F}} : \mathcal{F} \to \mathcal{F} \otimes_{\mathcal{O}_X} \Omega^1_X(\log B) \ , \tag{2.16.1}$$

which shows that the natural X/k-connection with poles along B on the locally free $\mathcal{O}_X$-module of rank d, $\mathcal{F} = \varphi_* \mathcal{O}_Y$, admits logarithmic singularities along B.

3 Semistable models

3.1. As a matter of notation, we recall that, for any k°-formal scheme $\mathfrak{X}$, locally of finite presentation, with generic fiber the k-analytic space $X = \mathfrak{X}_\eta$, there is a canonical *specialization map*

$$\mathrm{sp}_{\mathfrak{X}} : X = \mathfrak{X}_\eta \to \mathfrak{X}_s \ , \tag{3.1.1}$$

which may be viewed as a morphism of G-ringed spaces

$$\mathrm{sp}_{\mathfrak{X}} : X = X_G \to \mathfrak{X} \ , \tag{3.1.2}$$

where the subscript $(-)_G$ refers to the G-topology of [4, 1.3].

3.2. A preliminary step in the proof of theorem 0.1 is the understanding of the structure of the étale covering $\varphi^{-1}(E) \to E$, where E is the open disk $\varphi(D)$. This is the content of theorem 0.8. Its proof will be based on proposition 1.11, *i.e.* on the excellent convergence properties of the connection $(\mathcal{E}, \nabla_{\mathcal{E}})$ defined in (1.9.1). This will follow from an integrality statement for $(\mathcal{E}, \nabla_{\mathcal{E}})$ and from the transfer theorem 2.15 . Once this is established, we will prove theorem 0.1 by applying the trivial estimate 2.13 and the transfer theorem 2.15 to the connection $(\mathcal{F}, \nabla_{\mathcal{F}})$ defined in (1.2.2) and (2.16.1). In order to understand the integrality properties of $(\mathcal{F}, \nabla_{\mathcal{F}})$ and $(\mathcal{E}, \nabla_{\mathcal{E}})$, we discuss continuation of φ to a morphism of strictly semistable formal models of Y and X. We refer to [2, 1.1.4] for the relevant definitions.

3.3. We admit the following theorem (see [10], [21], [26], for similar statements in the algebraic setting).

Theorem 3.4. *Let $\mathfrak{Y}'$ (resp. $\mathfrak{X}'$) be any semistable k°-formal scheme with $\mathfrak{Y}'_\eta = Y$ (resp. $\mathfrak{X}'_\eta = X$). The map $\varphi : Y \to X$ admits a continuation to a finite morphism of strictly semistable k°-formal schemes $\Phi : \mathfrak{Y} \to \mathfrak{X}$ such that there exists an admissible blow-up $\mathfrak{Y} \to \mathfrak{Y}'$ (resp. $\mathfrak{X} \to \mathfrak{X}'$), and $\varphi = \Phi_\eta : \mathfrak{Y}_\eta \to \mathfrak{X}_\eta$. Moreover, we may assume that the divisors $Z \subset Y$ and $B \subset X$ extend to divisors $\mathfrak{Z} \subset \mathfrak{Y}^{\mathrm{sm}}$ and $\mathfrak{B} \subset \mathfrak{X}^{\mathrm{sm}}$ of the maximal smooth open formal k°-subscheme $\mathfrak{Y}^{\mathrm{sm}}$ of $\mathfrak{Y}$ and $\mathfrak{X}^{\mathrm{sm}}$ of $\mathfrak{X}$, respectively, both étale over k°, such that Φ induces a finite covering $\mathfrak{Z} \to \mathfrak{B}$.*

Remark 3.5. The map Φ is locally free of finite rank over $\mathfrak{X}^{\mathrm{sm}}$. This may be proven as in the classical algebraic case of a DVR R and of a finite morphism of R-semistable schemes of relative dimension 1 [20, Ex. 8.2.15]. In general, Φ cannot be chosen to be flat (unless one gives up finiteness by further admissible blowing-up, as in[9, Thm. 5.2]).

We refer to [2] for the correspondence $\mathfrak{Y} \leftrightarrow \mathbf{S}(\mathfrak{Y}) = \Gamma_{\mathfrak{Y}}$ (resp. $(\mathfrak{Y}, \mathfrak{Z}) \leftrightarrow \mathbf{S}_{\mathfrak{Z}}(\mathfrak{Y}) = \Gamma_{\mathfrak{Y},\mathfrak{Z}}$ between strictly semistable k°-formal models of X (resp. with a marked étale divisor $\mathfrak{Z} \subset \mathfrak{X}^{\mathrm{sm}}$) and connected metrized subgraphs of X, with edges of finite (resp. of both finite and infinite) length. This correspondence will also be shortly recalled in section 4 below.

3.6. We fix an extension $\Phi : \mathfrak{Y} \to \mathfrak{X}$ of $\varphi : Y \to X$, and the divisors $\mathfrak{Z} \subset \mathfrak{Y}$ and $\mathfrak{B} \subset \mathfrak{X}$, as in theorem 3.4.

We refer to [18] for the definitions and the differential constructions related to log-schemes. In particular, let Σ be the formal log-scheme, of the type discussed in [18, 1.5], supported by the formal scheme Spf k°, whose logarithmic structure is induced by the special fiber. As in *loc.cit.*, the pair $(\mathfrak{Y}, \mathfrak{Y}_s \cup \mathfrak{Z})$ determines a formal log-scheme over Σ, of underlying formal scheme $\mathfrak{Y}$, whose log-structure is given by the Zariski sheaf $\mathfrak{U} \mapsto \mathcal{M}(\mathfrak{U})$, where, for any open subset $\mathfrak{U} \subset \mathfrak{Y}$,

$$\mathcal{M}(\mathfrak{U}) = \{ g \in \mathcal{O}_{\mathfrak{Y}}(\mathfrak{U}) \,|\, g \text{ is invertible outside } \mathfrak{Y}_s \cup \mathfrak{Z} \} \ .$$

Although $(\mathfrak{Y}, \mathfrak{Y}_s \cup \mathfrak{Z})$ is not log-smooth over Σ, according to the definitions of [18], the sheaf of logarithmic differentials [18, 1.7] $\omega^1_{(\mathfrak{Y},\mathfrak{Y}_s\cup\mathfrak{Z})/\Sigma}$ is a sheaf of locally free $\mathcal{O}_{\mathfrak{Y}}$-modules of rank 1, canonically dual to the $\mathcal{O}_{\mathfrak{Y}}$-module $\mathcal{D}er((\mathfrak{Y}, \mathfrak{Z})/k^\circ)$ of k°-linear continuous derivations of $\mathcal{O}_{\mathfrak{Y}}$, which preserve the ideal sheaf of $\mathfrak{Z} \cup \mathfrak{Y}_s$. It is equipped with a canonical derivation

$$d_{(\mathfrak{Y},\mathfrak{Y}_s\cup\mathfrak{Z})/\Sigma} : \mathcal{O}_{\mathfrak{Y}} \longrightarrow \omega^1_{(\mathfrak{Y},\mathfrak{Y}_s\cup\mathfrak{Z})/\Sigma} .$$

Moreover, $\mathrm{sp}_{\mathfrak{Y}}^{-1}\omega^1_{(\mathfrak{Y},\mathfrak{Y}_s\cup\mathfrak{Z})/\Sigma}$ is a G-subsheaf of $\Omega^1_{Y_G/k}(\log Z)$ and

$$\Omega^1_{Y_G/k}(\log Z) = \mathrm{sp}_{\mathfrak{Y}}^{*}\omega^1_{(\mathfrak{Y},\mathfrak{Y}_s\cup\mathfrak{Z})/\Sigma} = k \otimes_{k^\circ} \mathrm{sp}_{\mathfrak{Y}}^{-1}\omega^1_{(\mathfrak{Y},\mathfrak{Y}_s\cup\mathfrak{Z})/\Sigma} . \tag{3.6.1}$$

Similarly for $(\mathfrak{X}, \mathfrak{X}_s \cup \mathfrak{B})$.

3.7. Since

$$\Phi(\mathfrak{Y} \setminus (\mathfrak{Y}_s \cup \mathfrak{Z})) \subset \mathfrak{X} \setminus (\mathfrak{X}_s \cup \mathfrak{B}) ,$$

the map Φ induces a finite morphism of log-schemes over Σ,

$$\Phi : (\mathfrak{Y}, \mathfrak{Y}_s \cup \mathfrak{Z}) \to (\mathfrak{X}, \mathfrak{X}_s \cup \mathfrak{B}) .$$

We will set

$$\mathfrak{F} := \Phi_*\mathcal{O}_{\mathfrak{Y}} \quad , \quad \mathfrak{E} := \Phi^*\Phi_*\mathcal{O}_{\mathfrak{Y}} .$$

Then $\mathfrak{F}$ (resp. $\mathfrak{E}$) is coherent $\mathcal{O}_{\mathfrak{X}}$-module (resp. $\mathcal{O}_{\mathfrak{Y}}$-module) such that $\mathfrak{F}_\eta = \mathcal{F}$ (resp. $\mathfrak{E}_\eta = \mathcal{E}$).

Remark 3.8. As we pointed out in remark 3.5, Φ is in general not flat over the singular locus of $\mathfrak{X}$. So, neither $\mathfrak{F}$ nor $\mathfrak{E}$ can be assumed to be locally free.

3.9. Let $\zeta_1, \dots, \zeta_t \in Y$ be the distinct φ-conjugates of $\zeta = \zeta_1$. Without loss of generality, we may assume that the points ζ_i, for $i = 1, \dots, t$ are vertices of $\Gamma_{\mathfrak{Y}}$ (resp. that $\xi = \varphi(\zeta) \in X$ is a vertex of $\Gamma_{\mathfrak{X}}$). In other words, ζ_i (resp. ξ) is the generic point on Y (resp. on X) of a component $\mathcal{D}'_i$ of $\mathfrak{Y}_s$ (resp. $\mathcal{C}$ of $\mathfrak{X}_s$). We consider the image by the specialization map $\mathrm{sp}_{\mathfrak{X}}$ of the open disk $E \subset X$. There are two possibilities:

1. $\mathrm{sp}_{\mathfrak{X}}(E)$ is a singleton in $\mathcal{C}^{\mathrm{sm}}$;
2. $\mathrm{sp}_{\mathfrak{X}}(E)$ contains a double point Q of $\mathcal{C}^{\mathrm{sm}}$.

On the other hand $\mathfrak{B}_s$ is a divisor in $\mathfrak{X}_s^{\mathrm{sm}}$, and the fact that $E \cap B = \mathfrak{B}_\eta = \emptyset$, guarantees that if we replace $\mathfrak{X}$ by $\mathfrak{X} \setminus \mathfrak{B}$ (resp. $\mathfrak{Y}$ by $\mathfrak{Y} \setminus \mathfrak{Z}$), Φ by its restriction to a finite map $\mathfrak{Y} \setminus \mathfrak{Z} \to \mathfrak{X} \setminus \mathfrak{B}$, and $\varphi : Y \to X$ by the generic fiber of the latter, we may assume from the beginning that $Z(\varphi) = \emptyset = B(\varphi)$, *i.e.* that φ is an étale covering.

3.10. We now reconsider the previous logarithmic complexes under the assumption that $\mathfrak{Z} = \emptyset = \mathfrak{B}$. So, $\omega^1_{(\mathfrak{Y},\mathfrak{Y}_s)/\Sigma}$ is a sheaf of locally free $\mathcal{O}_{\mathfrak{Y}}$-modules of rank 1, canonically dual to the $\mathcal{O}_{\mathfrak{Y}}$-module $\mathcal{D}er(\mathfrak{Y}/k^\circ)$ of k°-linear continuous derivations of $\mathcal{O}_{\mathfrak{Y}}$. Moreover, $\mathrm{sp}_{\mathfrak{Y}}^{-1}\omega^1_{(\mathfrak{Y},\mathfrak{Y}_s)/\Sigma}$ is a G-subsheaf of $\Omega^1_{Y_G/k}$ and

$$\Omega^1_{Y_G/k} = \mathrm{sp}_{\mathfrak{Y}}^{*}\omega^1_{(\mathfrak{Y},\mathfrak{Y}_s)/\Sigma} = k \otimes_{k^\circ} \mathrm{sp}_{\mathfrak{Y}}^{-1}\omega^1_{(\mathfrak{Y},\mathfrak{Y}_s)/\Sigma} . \tag{3.10.1}$$

Similarly for $(\mathfrak{X}, \mathfrak{X}_s)$.

3.11. Propositions 1.7 and 1.8 have the following general formal analogue, whose proof is essentially identical to the one of its algebraic form.

Proposition 3.12. *Let $\Phi : \mathfrak{Y} \to \mathfrak{X}$ be a finite morphism of k°-formal schemes locally of finite type. Let $\mathfrak{G}$ be any coherent $\mathcal{O}_{\mathfrak{Y}}$-module. Then*

$$\Phi^*\Phi_*\mathfrak{G} = \Phi^*\Phi_*\mathcal{O}_{\mathfrak{Y}} \otimes_{\mathcal{O}_{\mathfrak{Y}}} \mathfrak{G} \ .$$

Based on (3.12), we can now prove

Proposition 3.13. *The connection $(\mathcal{E}, \nabla_{\mathcal{E}})$ of (1.9.1) prolongs to a log-stratification on $\mathfrak{E}$.*

Proof. We use the standard notation of differential calculus [7]. The logarithmic version is developed in [25]. We need little more than notation from the previous sources. In particular, for $n = 1, 2, \dots$, we denote by $P^n_{Y/k} \rightrightarrows Y$ the analytic space of Y/k-jets of order n, equipped with its two finite free projections pr_1 and pr_2, to Y. The structure sheaf $\mathcal{P}^n_{Y/k}$ of $P^n_{Y/k}$, then has two $\mathcal{O}_Y$-module structures, for which we adopt the conventions of [7]. Similarly, we have $P^n_{(\mathfrak{Y},\mathfrak{Y}_s)/\Sigma} \rightrightarrows \mathfrak{Y}$ and $\mathcal{P}^n_{(\mathfrak{Y},\mathfrak{Y}_s)/\Sigma}$, where $(P^n_{(\mathfrak{Y},\mathfrak{Y}_s)/\Sigma})_\eta = P^n_{Y/k}$. The map $\varphi : Y \to X$ (resp. $\Phi : \mathfrak{Y} \to \mathfrak{X}$) induces maps $P^n(\varphi) : P^n_{Y/k} \to P^n_{X/k}$ (resp. $P^n(\Phi) : P^n_{(\mathfrak{Y},\mathfrak{Y}_s)/\Sigma} \to P^n_{(\mathfrak{Y},\mathfrak{Y}_s)/\Sigma}$), for any n. Let

(3.13.1) $$\varepsilon_{n,Y} : \mathcal{P}^n_{Y/k} \longrightarrow \mathcal{P}^n_{Y/k} \ ,$$

for $n = 1, 2, \dots$, be the canonical stratification on Y/k, and let

(3.13.2) $$\varepsilon_{n,\mathcal{E}} : \mathcal{P}^n_{Y/k} \otimes \mathcal{E} \longrightarrow \mathcal{E} \otimes \mathcal{P}^n_{Y/k} \ ,$$

be the stratification associated to $(\mathcal{E}, \nabla_{\mathcal{E}})$. We obviously have

(3.13.3) $$\varepsilon_{n,\mathcal{E}} = P^n(\varphi)^* P^n(\varphi)_* \varepsilon_{n,Y} \ ,$$

for any n. Similarly, let

(3.13.4) $$\varepsilon_{n,\mathfrak{Y}} : \mathcal{P}^n_{(\mathfrak{Y},\mathfrak{Y}_s)/\Sigma} \longrightarrow \mathcal{P}^n_{(\mathfrak{Y},\mathfrak{Y}_s)/\Sigma} \ ,$$

for $n = 1, 2, \dots$, be the canonical stratification on $(\mathfrak{Y}, \mathfrak{Y}_s)/\Sigma$. We now set $\mathfrak{E} := \Phi^*\Phi_*\mathcal{O}_{\mathfrak{Y}}$. For any n, Φ induces a finite morphism

(3.13.5) $$P^n(\Phi) : P^n_{(\mathfrak{Y},\mathfrak{Y}_s)/\Sigma} \to P^n_{(\mathfrak{Y},\mathfrak{Y}_s)/\Sigma} \ .$$

By proposition 3.12, we have
(3.13.6)
$$P^n(\Phi)^* P^n(\Phi)_* (\varepsilon_{n,\mathfrak{Y}} : \mathcal{P}^n_{(\mathfrak{Y},\mathfrak{Y}_s)/\Sigma} \to \mathcal{P}^n_{(\mathfrak{Y},\mathfrak{Y}_s)/\Sigma}) =: \varepsilon_{n,\mathfrak{E}} : \mathcal{P}^n_{(\mathfrak{Y},\mathfrak{Y}_s)/\Sigma} \otimes \mathfrak{E} \to \mathfrak{E} \otimes \mathcal{P}^n_{(\mathfrak{Y},\mathfrak{Y}_s)/\Sigma} \ ,$$

for any n. Since, obviously, $\varepsilon_{n,\mathcal{E}} = \mathrm{sp}^*_{P^n_{(\mathfrak{Y},\mathfrak{Y}_s)/\Sigma}}(\varepsilon_{n,\mathfrak{E}})$, for any n, we conclude that $\{\varepsilon_{n,\mathfrak{E}}\}_n$ is the desired extension of $\{\varepsilon_{n,\mathcal{E}}\}_n$ to a log-stratification of $\mathfrak{E}$. $\square$

Proof. (of proposition 1.11) $\square$

We recall [2] that, for any $y \in Y(k)$, the $\mathfrak{Y}$-unit open disk $D_{\mathfrak{Y}}(x, 1^-)$ centered at x, is the maximal open disk $C \subset Y$, such that $y \in C$, and $\mathrm{sp}_{\mathfrak{Y}}(C)$ is a singleton in $\mathfrak{Y}_s$. The open disk $D_{\mathfrak{Y}}(x, 1^-)$ is necessarily strict with maximal point $\tau_{\mathfrak{Y}}(y) \in \Gamma_{\mathfrak{Y}}$ of type 2.

Corollary 3.14. *For any $y \in Y(k)$, the solutions of $(\mathcal{E}, \nabla_{\mathcal{E}})$ converge and are bounded on the full disk $D_{\mathfrak{Y}}(x, 1^-)$.*

Proof. Let $D := D_{\mathfrak{Y}}(x, 1^-)$ and $\zeta := \tau_{\mathfrak{Y}}(y)$. Since $\mathcal{E}$ is a locally free $\mathcal{O}_Y$-module, it follows from corollary 2.9 that $(\mathscr{H}_Y(\mathcal{E}, D), || \ ||_D)$ is defined and that it is a free $\mathscr{H}_Y(D)$-module of rank d. The fact that $\nabla_{\mathcal{E}}$ continues to a log-stratification of $\mathfrak{E}$ means that, in terms of an overconvergent formal coordinate on D in Y, and of an $\mathscr{H}_Y(D)$-basis of $\mathscr{H}_Y(\mathcal{E}, D)$, the Taylor series of $\nabla_{\mathcal{E}}$ has coefficients of bounded $|| \ ||_D$-value. This proves the statement. $\square$

Proof. (of proposition 1.11). We now show how to deduce proposition 1.11 from corollary 3.14. For a sufficiently refined choice of $\mathfrak{Y}$ and of $\mathfrak{X}$ in theorem 3.4, we may assume that the boundary point ζ of the disk D appearing in the statement (1.11) in Y is a vertex of $\Gamma_{\mathfrak{Y}}$, *i.e.* the generic point in Y of a connected component of $\mathfrak{Y}_s$. If D is of the form $D_{\mathfrak{Y}}(y, 1^-)$, for some $y \in Y(k)$, the statement (1.11) holds true for D, by corollary 3.14. But the disk D is not necessarily of the form $D_{\mathfrak{Y}}(x, 1^-)$. It may in fact contain in its interior a nontrivial part of the graph $\Gamma_{\mathfrak{Y}}$. Still, we can consider the free $(\mathscr{H}_Y(\mathcal{E}, D), || \, ||_D)$-module of finite rank $(\mathscr{H}_Y(\mathcal{E}, D), || \, ||_D)$. The connection $\nabla_{\mathcal{E}}$, which has no singularities as a connection on the vector bundle $\mathcal{E}$, may show some apparent singularities in D, when it is expressed in terms of a particular $\mathscr{H}_Y(D)$-basis $\underline{v}$ of $\mathscr{H}_Y(\mathcal{E}, D)$. So, the differential system Σ for the vector of coordinates of a solution of $\nabla_{\mathcal{E}}$, expressed in terms of the basis $\underline{v}$, is of the form (2.1.2), where the coefficients of G are in the quotient field of $\mathscr{H}_Y(D)$ (but singularities in D are apparent). Moreover, corollary 3.14 shows that for any disk of the form $D' = D_{\mathfrak{Y}}(y, 1^-)$, with the same maximal point ζ, the corresponding differential system Σ' satisfies $R_{D' \subset Y}(\Sigma') = 1$. So, in the intrinsic notation of [2] or of [19], recalled in section 4, we have

$$\mathcal{R}_{\mathfrak{Y}}(\zeta, (\mathcal{E}, \nabla)) = IR(\mathcal{E}_{(\zeta)}, \nabla) = 1 \, .$$

But this coincides with $R_{D \subset Y}(\Sigma)$, independently of whether D is or is not of the form $D_{\mathfrak{Y}}(y, 1^-)$, for some $y \in Y(k)$. So, $R_{D \subset Y}(\Sigma) = 1$, and we may apply the transfer theorem 2.15, to deduce that the solutions of Σ are meromorphic on D. □

This proves proposition 1.11 and therefore also theorem 0.8.

3.15. Notice that in our discussion, we have the freedom to replace Y by any compact analytic domain U in Y containing D, provided the map φ restricts to a finite morphism $U \to \varphi(U)$.

Lemma 3.16. *Let notation be as in theorem 0.8, in particular $D \subset Y$ and $E = \varphi(D)$, and assume $\varphi^{-1}(E) \to E$ is étale. Then, there is a compact connected analytic domain U of Y containing $D \cup \{\zeta\}$, such that the restriction of φ to $U \to \varphi(U)$ is finite and is such that ζ is the unique inverse image of $\xi = \varphi(\zeta)$.*

Proof. Let $\zeta = \zeta_1, \dots, \zeta_r$ be the (distinct) points of Y which are *φ-conjugate to ζ*, *i.e.* such that $\varphi(\zeta_i) = \xi \in X$, for $i = 1, \dots, r$. Theorem 0.8 asserts that $\varphi^{-1}(E) \to E$ is a Galois covering, decomposed in a disjoint union of a finite number of (conjugate) Galois coverings of the form $D \to E$. So, for each $i = 1, \dots, r$, let $D_{i,1}, D_{i,1}, \dots, D_{i,s_i}$ be the distinct open disks, with boundary point ζ_i in Y, such that $D_{i,\ell} \to E$ is a Galois covering, which is a component of $\varphi^{-1}(E) \to E$.

Now, $\xi \notin \varphi(D)$, since $\varphi(D)$ is not compact; it follows that no ζ_i belongs to D. For $i = 1, \dots, r$, let U_i be a compact neighborhood of ζ_i. We assume that the U_i's are disjoint. Then, $D \cap U_1 \neq \emptyset$, but, for any $i \neq 1$, we have $D \cap U_i = \emptyset$. Otherwise, we would also have $\zeta_1 \in U_i$, contradicting disjointness. Let V be a connected affinoid neighborhood of ξ, such that $V \subset \varphi(U_i)$ for all i: such a V exists because φ, being flat, is open. For a sufficiently small V, we have $\varphi^{-1}(V) \subset \bigcup_{i=1}^r U_i$: otherwise, there would exist a sequence $\{y_h\}_{h=1,2,\dots}$, $y_h \in Y \setminus \bigcup_{i=1}^r U_i$, with $\varphi(y_h) \to \xi$, as $h \to \infty$. Notice that y_h belongs to the compact set $C := Y \setminus \bigcup_{i=1}^r \mathrm{Int}_Y(U_i)$, so that we may replace $\{y_h\}_{h=1,2,\dots}$, by a subsequence converging to $\zeta' \in C$. But then $\varphi(\zeta') = \xi$ would contradict the assumption that $\zeta_1, \dots, \zeta_r$ are all the points of Ywhich are φ-conjugate to $\zeta = \zeta_1$. So, after replacing U_i by $U_i \cap \varphi^{-1}(V)$, for $i = 1, \dots, r$, the restriction of φ to $\varphi_i : U_i \to V$ is finite, for any i. Notice that $\overline{U}_1 := U_1 \cup D$ is a connected compact neighborhood of ζ disjoint from U_i, for $i > 1$. Let $\overline{V} := \varphi(\overline{U}_1) = \varphi(U_1) \cup E$, a compact connected neighborhood of ξ containing V. The inverse image $\varphi^{-1}(\overline{V})$ is the disjoint union of r disjoint connected components $\overline{U}_1, \dots, \overline{U}_r$, where,

for any i,

$$\overline{U}_i = U_i \cup \bigcup_{\ell=1}^{s_i} D_{i,\ell} \ .$$

Notice that the component $\overline{U}_i$ is then a compact connected analytic domain which is a neighborhood of ζ_i and such that the restriction of φ to $\overline{\varphi}_i : \overline{U}_i \to \overline{V}$ is finite, for any i. Then $U = \overline{U}_1$ satisfies to the requirements in our statement. $\square$

So, without loss of generality, we make in the sequel the following

Assumption 3.17. *The morphism $\varphi : Y \to X$ of Theorem 0.1 satisfies the further condition that the boundary point ζ of D in Y is unique in its φ-conjugacy class and that $\xi := \varphi(\zeta) \notin \varphi(D)$.*

Remark 3.18. Under assumption 3.17, the image $E = \varphi(D)$ in X is an open disk with boundary point ξ.

3.19. We will now prove theorem 0.1. We will assume (3.17). We now come back to our choice of $\Phi : \mathfrak{Y} \to \mathfrak{X}$, as in theorem 3.4. We want to show that, by suitably restricting the source and the target spaces of $\varphi : Y \to X$, we may assume, without loss of generality, that $\mathfrak{Y}$ and $\mathfrak{X}$ are smooth, so that Φ will be (finite and) locally free, and that φ is an étale covering. We may assume, without loss of generality, that the point $\zeta \in Y$ (resp. $\xi = \varphi(\zeta) \in X$) is the generic point on Y (resp. on X) of a smooth component $\mathcal{C}'$ of $\mathfrak{Y}_s$ (resp. $\mathcal{C}$ of $\mathfrak{X}_s$), and that Φ induces a finite morphism of smooth $\widetilde{k}$-curves $\Phi_{\mathcal{C}',\mathcal{C}} : \mathcal{C}' \to \mathcal{C}$. As explained in the proof of lemma 3.16, let $D = D_1, \dots, D_r$ be the distinct open disks, each with boundary point ζ by (3.17), , which are the connected components of $\varphi^{-1}(E)$, $E = \varphi(D)$. We pick an affinoid with good reduction $U \subset \mathrm{sp}_{\mathfrak{X}}^{-1}(\mathcal{C})$ and maximal point ξ, such that φ restricts to a finite unramified map $U' := \varphi^{-1}(U) \to U$ and that U' itself has good reduction (and maximal point ζ). This is possible because $\mathfrak{B}_s \cap \mathcal{C}$ and $\mathfrak{Z}_s \cap \mathcal{C}'$ are finite sets of $\widetilde{k}$-rational points. Then (by possibly choosing a smaller U)

$$C' := U' \cup \bigcup_{i=1}^{r} D_i \quad , \quad C := U \cup E \quad , \tag{3.19.1}$$

are both affinoid domains with good canonical reduction in Y and X, respectively, such that φ induces a finite unramified map $\varphi : C' \to C$. In this situation the map φ extends to a finite map $\Phi : \mathfrak{C}' \to \mathfrak{C}$ of the smooth model $\mathfrak{C}'$ of C' to the smooth model $\mathfrak{C}$ of C. Notice that Φ is automatically flat.

3.20. Let us consider the assumption

Assumption 3.21. *The map $\varphi : Y \to X$ is unramified and it is the generic fiber of a finite free morphism $\Phi : \mathfrak{Y} \to \mathfrak{X}$ of smooth affine formal schemes, both étale over the formal affine line over k°, where the open disk D is a residue class of $\mathfrak{Y}$.*

We conclude

Proposition 3.22. *It suffices to prove theorem 0.1 and proposition 1.11 under the further assumption 3.21.*

Remark 3.23. If φ is residually separable at the maximal point ζ of Y, the morphism Φ of (3.21) is then an étale covering. So in this case, under the assumption 3.21, theorem 0.1 follows from [5, Lemma 2.2]: in fact in this case ϕ induces an isomorphism between any residue class in Y and its image in X.

3.24. We assume here (3.21). Let $\mathfrak{F} := \Phi_* \mathcal{O}_{\mathfrak{Y}}$, a coherent and locally free $\mathcal{O}_{\mathfrak{X}}$-module such that $\mathfrak{F}_\eta = \mathcal{F}$.

Proposition 3.25. *There exists $\pi_\Phi \in k^{\circ\circ}$, non-zero, such that*

$$\Phi^* \Omega^1_{\mathfrak{X}/k^\circ} = \pi_\Phi \Omega^1_{\mathfrak{Y}/k^\circ} \ .$$

Therefore

$$\Phi_* \Omega^1_{\mathfrak{Y}/k^\circ} = \mathfrak{F} \otimes \pi_\Phi^{-1} \Omega^1_{\mathfrak{X}/k^\circ} \ .$$

Proof. The second statement follows from the first by the projection formula. Let $\xi_{\mathfrak{Y}}$ (resp. $\xi_{\mathfrak{X}}$) be the generic point of $\mathfrak{Y}$ (resp. $\mathfrak{X}$), and let ξ_Y (resp. ξ_X) be the maximal point of Y (resp. X). The local ring $\mathcal{O}_{\mathfrak{Y},\xi_{\mathfrak{Y}}}$ (resp. $\mathcal{O}_{\mathfrak{X},\xi_{\mathfrak{X}}}$) of $\xi_{\mathfrak{Y}}$ (resp. $\xi_{\mathfrak{X}}$) is a valuation ring of rank 1: its valuation extends the one of k°, and has the same value group. We have $\mathcal{O}_{\mathfrak{Y},\xi_{\mathfrak{Y}}} = \kappa(\xi_Y)^\circ$ (resp. $\mathcal{O}_{\mathfrak{X},\xi_{\mathfrak{X}}} = \kappa(\xi_X)^\circ$), hence $k \otimes_{k^\circ} \mathcal{O}_{\mathfrak{Y},\xi_{\mathfrak{Y}}} = \mathcal{O}_{Y,\xi_Y} = \kappa(\xi_Y)$ (resp. $k \otimes_{k^\circ} \mathcal{O}_{\mathfrak{X},\xi_{\mathfrak{X}}} = \mathcal{O}_{X,\xi_X} = \kappa(\xi_X)$). Let $\pi_\Phi \in k^{\circ\circ}$ be such that

$$(\Phi^* \Omega^1_{\mathfrak{X}/k^\circ})_{\xi_{\mathfrak{Y}}} = \pi_\Phi (\Omega^1_{\mathfrak{Y}/k^\circ})_{\xi_{\mathfrak{Y}}} \ .$$

Let E be any maximal open disk in Y. Since $\varphi(\xi_Y) = \xi_X$, $E' := \varphi(E)$ is a maximal open disk in X. Let T (resp. S) be a formal coordinate on E in Y (resp. on E' in X). The map φ is then expressed in E by

$$S = h(T) \ ,$$

where $h(T) \in k[[T]]$ is a power series converging and bounded in E, with $||h||_E = |h(\xi_Y)| = 1$. Since φ is unramifed on E, the derivative dh/dT does not vanish on E, hence it has a constant absolute value, necessarily equal to $|\pi_\Phi|$. $\square$

Corollary 3.26. *For any $b \in X(k)$, the connection $(\mathcal{F}, \nabla_{\mathcal{F}})$ admits a full set of solutions converging in $D_{\mathfrak{X}}(b, p^{-\frac{1}{p-1}}|\pi_\Phi|^-)$.*

Proof. It is an immediate consequence of the trivial estimate (2.13) and of the transfer theorem (2.15). $\square$

Remark 3.27. The constant $-\frac{1}{p-1} + \mathrm{ord}_p \pi_\Phi$ may be bound uniformly from below in terms of d and p. See [22, Thm. 2.1].

Corollary 3.28. *For any $a \in Y(k)$, the map φ restricts to an open immersion of $D_{\mathfrak{Y}}(a, (p^{-\frac{1}{p-1}})^-)$ in X.*

Proof. We consider a section $\sigma : D_b := D_{\mathfrak{X}}(b, p^{-\frac{1}{p-1}}|\pi_\Phi|^-) \to Y$ of $\varphi : Y \to X$, and let $a = \sigma(b)$. Then $\sigma(D_b) =: D_a$, is an open disk in Y, $a \in D_a$, and φ restricts to an isomorphism $\varphi_{a,b} : D_a \xrightarrow{\sim} D_b$. Let E (resp. E'), as in the proof of proposition 3.25 be a residue class of Y (resp. X) containing D_a (resp. D_b), and let us use the notation of *loc.cit.*; in particular, we have $|(dh/dT(y)| = |\pi_\Phi|$, for any $y \in Y$. The p-adic Newton lemma [13, I.4.2] implies that, for any $\varepsilon \in (0,1)$, and any $b_1 \in E'(k)$, with $|S(b_1) - S(b)| < \varepsilon |\pi_\Phi|^2$, there is a unique $a_1 \in E(k)$, with $|T(a_1) - T(a)| < \varepsilon |\pi_\Phi|$, such that $\varphi(a_1) = b_1$. So, for $a_1, a_2 \in D_a(k)$, and $b_1 = \varphi(a_1), b_2 = \varphi(a_2) \in D_b(k)$, if $|S(b_1) - S(b_2)| < |\pi_\Phi|^2$ and $|T(a_1) - T(a_2)| < |\pi_\Phi|$, we have

$$|T(a_1) - T(a_2)| \leq |\pi_\Phi|^{-1} |S(b_1) - S(b_2)| \ .$$

On the other hand, since $|(dh/dT(y)|$ has the constant value $|\pi_\Phi|$ on E, there exists $\varepsilon \in (0,1)$, such that, if $|T(a_1) - T(a_2)| < \varepsilon$, then

$$|S(b_1) - S(b_2)| \leq |\pi_\Phi| |T(a_1) - T(a_2)| \ .$$

In other words, there exists $\varepsilon \in (0,1)$, such that, if $|T(a_1) - T(a_2)| < \varepsilon$, then

$$|S(b_1) - S(b_2)| = |\pi_\Phi||T(a_1) - T(a_2)| \ . \tag{3.28.1}$$

Now, the map $\varphi_{a,b} : D_a \xrightarrow{\sim} D_b$ being an isomorphism, this estimate must hold for any $a_1, a_2 \in D_a(k)$, and for their images $b_1 = \varphi(a_1)$ and $b_2 = \varphi(a_2) \in D_b(k)$ [3, 6.4.4]. In particular,

$$D_a = D_{\mathfrak{Y}}(a, (p^{-\frac{1}{p-1}})^-) \ . \tag{3.28.2}$$

This proves the proposition. □

We have thus concluded the proof of Theorem 0.1.

4 Graphs and radius of convergence

4.1. We recall from [2] that to the pair $(\mathfrak{Y}, \mathfrak{Z})$ we can associate a subgraph $\Gamma_{(\mathfrak{Y},\mathfrak{Z})}$ of the profinite graph Y, equipped with a continuous retraction $\tau_{(\mathfrak{Y},\mathfrak{Z})} : Y \to \Gamma_{(\mathfrak{Y},\mathfrak{Z})}$. Notice that we are extending the graph $\Gamma_{(\mathfrak{Y},\mathfrak{Z})}$ of [2] to include the points of $Z \subset Y(k)$ as vertices "at infinite distance" and the retraction $\tau_{(\mathfrak{Y},\mathfrak{Z})} : Y \to \Gamma_{(\mathfrak{Y},\mathfrak{Z})}$ by $\tau_{(\mathfrak{Y},\mathfrak{Z})}(z) = z$, for any $z \in Z$. We do not exclude the case $\mathfrak{Z} = \emptyset$, and we sometimes write $\tau_{\mathfrak{Y}} : Y \to \Gamma_{\mathfrak{Y}}$ if this is the case. The fibers of the retraction $\tau_{(\mathfrak{Y},\mathfrak{Z})}$ over points of Berkovich type 2 are the closures in Y of the maximal open disks contained in $Y \setminus \Gamma_{(\mathfrak{Y},\mathfrak{Z})}$. Any such maximal open disk E contains at least a k-rational point $x \in Y(k)$; we define $E =: D_{(\mathfrak{Y},\mathfrak{Z})}(x, 1^-)$. As a k-analytic curve, $D_{(\mathfrak{Y},\mathfrak{Z})}(x, 1^-)$ is isomorphic to the standard open k-disk in $\mathbf{P}$, $D(0, 1^-)$, via a $(\mathfrak{Y}, \mathfrak{Z})$-*normalized coordinate at* x. Given any object $(\mathcal{M}, \nabla)$ of $\mathbf{MIC}((Y \setminus Z)/k)$, and any $x \in Y(k) \setminus Z$, we can define, as in [2], the $(\mathfrak{Y}, \mathfrak{Z})$-*normalized radius of convergence of* $(\mathcal{M}, \nabla)$ *at* x, $\mathcal{R}_{(\mathfrak{Y},\mathfrak{Z})}(x, (\mathcal{M}, \nabla))$, as the radius, measured in $(\mathfrak{Y}, \mathfrak{Z})$-normalized coordinate at x, of the maximal open disk E centered at x and contained in $Y \setminus \Gamma_{(\mathfrak{Y},\mathfrak{Z})}$, such that $(\mathcal{E}, \nabla)_{|E}$ is a free $\mathcal{O}_E$-module of finite rank, equipped with the trivial connection.

4.2. We can also extend the definition of $\mathcal{R}_{(\mathfrak{Y},\mathfrak{Z})}(x, (\mathcal{M}, \nabla))$ to the case when $x \in Y \setminus Z$ is not necessarily k-rational. In full generality, let K/k be a completely valued field extension, let $Y_K = Y \widehat{\otimes}_k K$ and let $\pi_{K/k} : Y_K \to Y$, be the projection. Then there is a canonical functor *change of field of constants by* K/k

$$\begin{array}{rccc} \pi^*_{K/k} : & \mathbf{MIC}((Y \setminus Z)/k) & \to & \mathbf{MIC}((Y_K \setminus Z)/K) \\ & (\mathcal{M}, \nabla) & \mapsto & \pi^*_{K/k}(\mathcal{M}, \nabla) \ . \end{array} \tag{4.2.1}$$

So, let $x \in Y \setminus Z$, not necessarily k-rational. As in [2], we change the field of constants by $\mathscr{H}(x)/k$, and pick (canonically) a $\mathscr{H}(x)$-rational point $x' \in Y \widehat{\otimes}_k \mathscr{H}(x)$ above x. We then set

$$\mathcal{R}_{(\mathfrak{Y},\mathfrak{Z})}(x, (\mathcal{M}, \nabla)) := \mathcal{R}_{(\mathfrak{Y}\widehat{\otimes}_{k^\circ}\mathscr{H}(x)^\circ, \mathfrak{Z}\widehat{\otimes}_{k^\circ}\mathscr{H}(x)^\circ)}(x', \pi^*_{\mathscr{H}(x)/k}(\mathcal{M}, \nabla)) \ . \tag{4.2.2}$$

This definition is compatible with any change of the field of constants by any K/k in the sense that, for any K/k and any $y \in Y_K \setminus Z$,

$$\mathcal{R}_{(\mathfrak{Y}\widehat{\otimes}_{k^\circ}K^\circ, \mathfrak{Z}\widehat{\otimes}_{k^\circ}K^\circ)}(y, \pi^*_{K/k}(\mathcal{M}, \nabla)) = \mathcal{R}_{(\mathfrak{Y},\mathfrak{Z})}(\pi_{K/k}(y), (\mathcal{M}, \nabla)) \ . \tag{4.2.3}$$

The function $x \mapsto \mathcal{R}_{(\mathfrak{Y},\mathfrak{Z})}(x, (\mathcal{M}, \nabla))$ is conjectured to be continuous on $Y \setminus Z$, for any $(\mathcal{M}, \nabla) \in \mathbf{MIC}((Y \setminus Z)/k)$. This conjecture was proven in [2] under the assumption that $(\mathcal{M}, \nabla) \in \mathbf{MIC}_{(\mathfrak{Y},\mathfrak{Z})}(X(*Z)/k)$, *i.e.* that $\mathcal{M}$ extends to a locally free coherent $\mathcal{O}_{\mathfrak{Y}}$-module and ∇ has meromorphic singularities at Z.

4.3. We now explain the difference between our radius of convergence $\mathcal{R}_{(\mathfrak{Y},\mathfrak{Z})}(x,(\mathcal{M},\nabla))$ and the *intrinsic radius of convergence* $IR(\mathcal{M}_{(x)},\nabla)$ of

$$(\mathcal{M}_{(x)},\nabla) := (\mathcal{M},\nabla)_x \otimes_{\mathcal{O}_{Y,x}} \mathscr{H}(x)\ , \tag{4.3.1}$$

for $x \in Y$ of Berkovich type 2 or 3, of Kedlaya [19, Def. 9.4.7]. Here $\mathcal{O}_{Y,x} = \kappa(x)$ is a valued field [4, 2.1], $(\mathcal{M},\nabla)_x$ is a $\kappa(x)/k$-differential module and $(\mathcal{M}_{(x)},\nabla)$ is its completion.[1] Both definitions go back to Dwork and Robba; the latter was refined by Christol-Dwork and used by Christol-Mebkhout and André. We will show that two notions coincide at the points $x \in \Gamma_{(\mathfrak{Y},\mathfrak{Z})} \setminus Z$.

4.4. Let us shortly review, in our own words, the definition of $IR(\mathcal{M}_{(x)},\nabla)$, taken from [19, Chap. 9]. Let $(F,|\ |_F)/(k,|\ |)$ be a complete extension field. Then $(F,|\ |_F)$ is a k-Banach algebra, and so is $\mathcal{L}_k(F)$, for the operator norm. Similarly, on a finite dimensional F-vector space M, all norms compatible with $|\ |_F$ are equivalent and define equivalent structures of k-Banach space on M. It will be understood in the following that any such M is given some norm of F-vector space, compatible with $|\ |_F$, and then $\mathcal{L}_k(M)$ is given the corresponding operator norm. The definitions will be independent of the choices made.

Under the previous assumptions $\mathcal{L}_k(F)$ (resp. $\mathcal{L}_k(M)$) will be regarded as an F-vector space via the *left* action, $(a\,L)(b) = a\,L(b)$, for $a,b \in F$ (resp. $a \in F$, $b \in M$) and $L \in \mathcal{L}_k(F)$ (resp. $\mathcal{L}_k(M)$) .

Definition 4.5. *A* complete differential field of dimension 1 over $(k,|\ |)$ *is a complete extension field* $(F,|\ |_F)/(k,|\ |)$ *such that the* F*-vector space* $Der(F/k) \subset \mathcal{L}_k(F)$ *of bounded* k*-linear derivations of* F*, is of dimension 1. A* based complete differential field (of dimension 1) over $(k,|\ |)$ *is a triple* $(F,|\ |_F,\partial)$ *where* $(F,|\ |_F)/(k,|\ |)$ *is a complete extension field and* $0 \neq \partial \in Der(F/k)$.

Example 4.6. A point $x \in \mathbf{P}$ of type 2 (resp. 3) is the point $t_{a,\rho}$ at the boundary of the open disk $D(a,\rho^-)$, for $a \in k$ and $\rho > 0$ in $|k|$ (resp. in $\mathbb{R} \setminus |k|$). One defines [19, Def. 9.4.1] $F_{a,\rho} = \mathscr{H}(x)$, as the completion of $k(T)$ under the absolute value

$$f(T) \mapsto |f|_{a,\rho} := |f(t_{a,\rho})|\ .$$

Let $\mathcal{L}_k(F_{a,\rho})$ be the k-Banach algebra of bounded k-linear endomorphisms of the k-Banach algebra $F_{a,\rho}$, equipped with the operator norm. We still denote the operator norm by $|\ |_{a,\rho}$. Then $\frac{d}{dT}$ extends by continuity to a k-derivation of $F_{a,\rho}$, and

$$|\frac{d}{dT}|_{a,\rho} = \rho^{-1}\ , \tag{4.6.1}$$

as an element of $\mathcal{L}_k(F_{a,\rho})$. For the spectral norm of $\frac{d}{dT} \in \mathcal{L}_k(F_{a,\rho})$, we have

$$|\frac{d}{dT}|_{\mathrm{sp},a,\rho} = p^{-\frac{1}{p-1}}\,\rho^{-1}\ . \tag{4.6.2}$$

So, the pair (resp. the triple) $(F_{a,\rho},|\ |_{a,\rho})$ (resp. $(F_{a,\rho},|\ |_{a,\rho},\frac{d}{dT})$) is a (resp. based) complete differential field of dimension 1 over $(k,|\ |)$.

Remark 4.7. Let $(F,|\ |_F)$ be a complete differential field of dimension 1 over $(k,|\ |)$. Then, for any F-basis ∂ of $Der(F/k)$ and for any $n \geq 0$, the F-vector subspace $Diff^n(F/k) \subset \mathcal{L}_k(F)$ of bounded k-linear differential operators of F of order $\leq n$, is freely generated by $\mathrm{id}_F,\partial,\ldots,\partial^n$.

[1] The reader should appreciate the difference between the operation $(\mathcal{M},\nabla)) \mapsto (\mathcal{M}_{(x)},\nabla)$, resulting in a $\mathscr{H}(x)/k$-differential module, and the change of field of constants by $\mathscr{H}(x)/k$, $(\mathcal{M},\nabla) \mapsto \pi^*_{\mathscr{H}(x)/k}(\mathcal{M},\nabla)$, resulting in an object of $\mathbf{MIC}((Y_{\mathscr{H}(x)} \setminus Z)/\mathscr{H}(x))$.

Definition 4.8. *A* finite dimensional differential module over the complete differential field *$(F, |\ |_F)$ (of dimension 1 over $(k, |\ |)$) is a pair (M, ∇) consisting of a finite dimensional F-vector space M and of a k-linear bounded F-algebra homomorphism*

$$\nabla : Diff(F/k) \to \mathcal{L}_k(M) ,$$

such that

$$\nabla(\partial)(a\, m) = \partial(a)\, m + a\, \nabla(\partial)(m) ,$$

for any $\partial \in Der(F/k)$, $a \in F$ and $m \in M$. If we specify a generator ∂ of $Der(F/k)$ and the corresponding $\Delta = \nabla(\partial)$, we obtain the based *finite dimensional differential module (M, Δ) over the based complete differential field $(F, |\ |_F, \partial)$.*

Remark 4.9. Conversely, given a based finite dimensional differential module (M, Δ) over the based complete differential field $(F, |\ |_F, \partial)$, one defines (M, ∇) by setting

$$\nabla(\sum_{i=0}^{n} a_i\, \partial^i) = \sum_{i=0}^{n} a_i\, \Delta^i ,$$

for any n and any $a_0, \dots, a_n \in F$. It is clear that ∇ is a bounded F-algebra homomorphism

$$\nabla : Diff(F/k) \to \mathcal{L}_k(M) .$$

Definition 4.10. *Let $(M, \nabla(\partial)) = (M, \Delta)$ be a nonzero finite dimensional based differential module over the based complete differential field $(F, |\ |_F, \partial)$. The* extrinsic radius of convergence *of (M, Δ) is*

$$R(M, \Delta) = p^{-\frac{1}{p-1}}\, |\Delta|_{\mathrm{sp}}^{-1} > 0 ,$$

where $|\Delta|_{\mathrm{sp}}$ is the spectral norm of Δ of the k-Banach algebra $\mathcal{L}_k(M)$.

Definition 4.11. *Let (M, ∇) be a finite dimensional differential module over the complete differential field $(F, |\ |_F)$. The* intrinsic radius of convergence *of (M, ∇) is*

$$IR(M, \nabla) = R(M, \nabla(\partial))\, p^{\frac{1}{p-1}}\, |\partial|_{\mathrm{sp}} = |\partial|_{\mathrm{sp}}\, |\Delta|_{\mathrm{sp}}^{-1} \in (0, 1] ,$$

for any non zero element $\partial \in Der(F/k)$.

The following proposition explains why $IR(M, \nabla)$ deserves the attribute *intrinsic*.

Proposition 4.12. *For any $n = 0, 1, \dots,$ let $c_n \in \mathbb{R}_{>0}$ be the operator norm of the map of k-Banach spaces*

$$\nabla_n = \nabla_{|Diff^n(F/k)} : Diff^n(F/k) \to \mathcal{L}_k(M) .$$

Then

$$IR(M, \nabla) = \liminf_{n \to \infty} c_n^{-1/n} . \tag{4.12.1}$$

Proof. Essentially follows from [19, Prop. 6.3.1]. ☐

Corollary 4.13. *Let $(\mathcal{M}, \nabla) \in$ **MIC**$((Y \setminus Z)/k)$, as before. Let $\zeta \in Y$ be a point of Berkovich type 2. Let D be any open disk in Y with boundary point ζ, let $T \in \mathcal{O}_{Y,\zeta}$ be the germ of a normalized coordinate on D*

$$T : D \xrightarrow{\ \sim\ } D(0, 1^-) .$$

For any $r \in (0, 1) \cap |k|$, let $C(0; r, 1)$ be as in (2.1.1). For r close to 1, we identify the restriction $(\mathcal{M}, \nabla)_{|C(0;r,1)}$, via the choice of a basis of sections of $\mathcal{M}$ in a neighborhood of ζ containing $C(0; r, 1)$, with a differential system Σ of the form 2.1.2. Then

$$IR(\mathcal{M}_{(\zeta)}, \nabla) = R_{D \subset Y}(\Sigma) . \tag{4.13.1}$$

Remark 4.14. We chose a point ζ of type 2, rather than allowing points of type 3 as well, only in order to avoid extending the field of definition k and to establish contact with the notation of (2.1.1).

Remark 4.15. In formula 4.12.1, no formal semistable model $\mathfrak{Y}$ of Y explicitly appears. Such a (smooth) model is hidden, however, in the absolute value corresponding to the point x of type 2 or 3. As explained in corollary 4.13, the normalization of measures at x of type 2 or 3 in this case varies with x and is obtained by taking as an open disk of radius 1, *any open disk with boundary point* x.

4.16. The disadvantage of the function $x \mapsto IR(\mathcal{M}_{(x)}, \nabla)$ which describes the intrinsic radius of convergence of $\mathcal{M}$ at $x \in Y$ of type 2 or 3, is that it cannot possibly be extended by continuity to $Y \setminus Z$. In fact, for any point $x_0 \in Y(k) \setminus Z$, one obviously has

$$\lim_{x \to x_0} IR(\mathcal{M}_{(x)}, \nabla) = 1 \ , \tag{4.16.1}$$

where the limit runs over the points x of type 2 or 3. But, $Y(k) \setminus Z$ is dense in $Y \setminus Z$, so one would have $R((\mathcal{M}, \nabla), x) = 1$ identically on $Y \setminus Z$, which is obviously not always the case. The point in our definition of the function $x \mapsto \mathcal{R}_{(\mathfrak{Y},\mathfrak{Z})}(x, (\mathcal{M}, \nabla))$ [2] is that

1. it interpolates the classical notion of radius of convergence, normalized by the choice of $(\mathfrak{Y}, \mathfrak{Z})$;

2. it is compatible with extension of the ground field;

3. it coincides on the graph $\Gamma_{(\mathfrak{Y},\mathfrak{Z})}$ with the intrinsic radius of convergence

$$IR(\mathcal{M}_{(x)}, \nabla) = \mathcal{R}_{(\mathfrak{Y},\mathfrak{Z})}(x, (\mathcal{M}, \nabla)) \ , \ \text{ if } \ x \in \Gamma_{(\mathfrak{Y},\mathfrak{Z})} \ . \tag{4.16.2}$$

The last property follows from remark 4.15.

References

[1] Yves André and Francesco Baldassarri : *De Rham Cohomology of Differential Modules on Algebraic Varieties*, Progress in Mathematics, volume 189, Birkhaüser Verlag 2001, 214 pages.

[2] Francesco Baldassarri : Continuity of the radius of convergence of differential equations on p-adic analytic curves. *Inventiones Math.* 182(3): 513–584, 2010.

[3] Vladimir G. Berkovich : *Spectral theory and analytic geometry over non-Archimedean fields*, volume 33 of *Mathematical Surveys and Monographs.* American Mathematical Society, Providence, RI, 1990.

[4] Vladimir G. Berkovich : Étale cohomology for non-Archimedean analytic spaces. *Institut des Hautes Études Scientifiques. Publications Mathématiques*, 78:5–161, 1993.

[5] Vladimir G. Berkovich. Vanishing cycles for formal schemes. *Inventiones Math.*, 115(3):539–571, 1994.

[6] Vladimir G. Berkovich. *Integration of one-forms on p-adic analytic spaces*, volume 162 of *Annals of Mathematics Studies.* Princeton University Press, Princeton, NJ, 2007.

[7] Pierre Berthelot and Arthur Ogus : *Notes on crystalline cohomology*, Mathematical Notes, Princeton University Press, 1978.

[8] Siegfried Bosch and Werner Lütkebohmert : Stable reduction and uniformization of abelian varieties. I. *Math. Ann.*, 270(3):349–379, 1985.

[9] Siegfried Bosch and Werner Lütkebohmert. Formal and rigid geometry. II. Flattening techniques. *Math. Ann.*, 296:403–429, 1993.

[10] Robert F. Coleman : Stable maps of curves. *Documenta Math.*, A collection of manuscripts written in honour of Kazuya Kato on the occasion of his fiftieth birthday. *Spencer Bloch et al. Eds.*, 217–225, 2003.

[11] Antoine Ducros : *La structure des courbes analytiques*, manuscript in preparation.

[12] Aise Johann de Jong : Étale fundamental groups of non-archimedean analytic spaces. *Compositio Mathematica*, 97:89–118, 1995.

[13] Bernard Dwork, Giovanni Gerotto, and Francis J. Sullivan : *An introduction to G-functions*, volume 133 of *Annals of Mathematics Studies*. Princeton University Press, 1994.

[14] Bernard Dwork, and Philippe Robba : On natural radii of p-adic convergence. *Transactions of the AMS*, 256 : 199–213, 1979.

[15] Xander Faber : Topology and geometry of the Berkovich ramification locus for rational functions. Preprint, arXiv:1102.1432v3 [math.NT], 2011.

[16] Xander Faber : Topology and geometry of the Berkovich ramification locus for rational functions II. Preprint, arXiv:1104.0943v2 [math.NT].

[17] Jean Fresnel, Michel Matignon. Sur les espaces analytiques quasi-compacts de dimension 1 sur un corps valué complet ultramétrique. *Annali di Matematica Pura e Applicata*, 145 (4): 159–210, 1986.

[18] Kazuya Kato : Logarithmic structures of Fontaine-Illusie. *Algebraic Analysis, Geometry, and Number Theory Igusa, J.-I., ed.*, Johns Hopkins Univ. Press, Baltimore, 1989, 191-224.

[19] Kiran S. Kedlaya : *p-adic Differential Equations*, volume 125 of Cambridge Studies in Advanced Mathematics, Cambridge Univ. Press, Cambridge, 2010.

[20] Qing Liu : *Algebraic Geometry and Arithmetic Curves*, Oxford Graduate Texts in Mathematics, volume 6 Oxford University Press, 2002.

[21] Qing Liu : Stable reduction of finite covers of curves. *Compositio Math.* 142: 101–118, 2006.

[22] Werner Lütkebohmert : Riemann's existence theorem for a p-adic field. *Inventiones Math.* 111: 309–330, 1993.

[23] Alain Robert : *A course in p-adic analysis*, Graduate Texts in Mathematics, vol. 198, Springer-Verlag, New York, 2000.

[24] Matthew Baker and Robert S. Rumely : *Potential theory on the Berkovich projective line*, Mathematical Surveys and Monographs, Vol. 159, AMS, 2010.

[25] Atsushi Shiho : Crystalline fundamental groups I - Isocrystals on log crystalline site and log convergent site. *J. Math. Sci. Univ. Tokyo* 7: 509–656, 2000.

[26] Michael Temkin : A new proof of the stable reduction theorem. PhD Thesis Weizmann Institute 2006, (partly) published as : Stable modification of relative curves. *J. Alg. Geom.* 19: 603–677, 2010.